Using perturbed resonant frequencies to study eigenmodes of an acoustic resonator

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A simple experiment is described which is based on the notion that the resonant frequency of an eigenmode is shifted if an obstacle is introduced into an acoustic resonator. The shift depends on the shape and volume of the obstacle and on the nature of the acoustic field which previously existed in the space occupied by the obstacle. It is shown that a spherical obstacle made from a set of parallel disks can be used to evaluate both the intensity and direction of motion of the vibrations in the resonator. The relevance of the method to other fields of physics is also indicated.

I. INTRODUCTION

The shift in resonant frequency caused by perturbing a resonant system is a powerful method of accurately measuring physical parameters. The principal reason for the popularity of the technique is that the errors are commonly limited by the accuracy with which one can measure frequencies; for high quality resonators the error limits can be very small. Examples of the technique can be found in any university level textbook on physics; microwave spectroscopy proves a particularly good illustration of the technique.

The purpose of the experiment described below is to provide the student with an inexpensive and reliable demonstration of the principles underlying this method of measurement. The experiment makes use of an acoustic rather than a microwave resonator. For conventional measurements using resonant frequency shifts, the structure of the eigenmode in the resonator is known and the frequency shift provides information about the properties of the obstacle introduced into it, whereas in the experiment described below a converse procedure is adopted: An obstacle of known properties is brought into the resonator and the shift of the resonant frequency is used to determine the properties of the eigenmode. The obstacle is essentially a rigid sphere, and as the theory below shows, the shift in resonant frequency provides information on the magnitude of the eigenmode at any point, but gives no indication of the direction of vibration. The latter information is obtained by making a slotted sphere from a set of parallel disks. The shift in the resonant frequency of the eigenmode is clearly minimized when the eigenmode vibrates parallel to the plane of the disks since in this configuration the disks hardly impede the vibrations at all.

The theoretical model which relates shifts in resonant frequency to the properties of the eigenmodes and the nature of the resonator perturbation is usually first encountered by the contemporary student when he studies the properties of microwave resonators. For acoustic resonators, however, the problem is much simpler than the more familiar one cited above, because acoustic modes are described by scalar fields and the boundary conditions are intuitively obvious. The theory for an acoustic resonator is therefore presented below, albeit in an abbreviated form, so that the article can be understood without the need to refer to other work.

The theory is followed by a description of the experiment and a selection of typical results. The paper is concluded by suggestions concerning the possibility of setting up a similar type of experiment using a microwave resonator instead of an acoustic one.

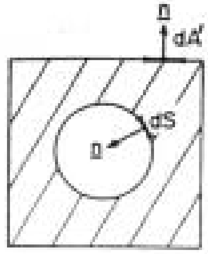


Fig. 1. Axial section of resonator through obstacle. The shaded area indicates the volume of the resonator.

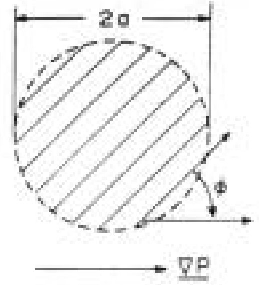


Fig. 2. Axial section of slotted spherical probe. P = pressure in sound wave, ϕ = angle between slots and ∇P, and a = radius of probe.

II. THEORY

Let P and P' denote the respective pressures in the resonator before and after the obstacle is introduced in it. The corresponding resonant frequencies are ω and $\omega + \Delta\omega$, where it is assumed that $\Delta\omega/\omega \ll 1$. It is also assumed that the pressures satisfy the wave equations

$$\nabla^2 P = -\omega^2 P/c^2, \tag{1}$$

$$\nabla^2 P' = -(\omega + \Delta\omega)^2 P'/c^2, \tag{2}$$

where c is the speed of sound.

These equations have to be solved subject to the boundary conditions

$$\mathbf{n} \cdot \nabla P = 0, \tag{3}$$

$$\mathbf{n} \cdot \nabla P' = 0 \tag{4}$$

in the original [Eq. (3)] and perturbed resonator [Eq. (4)], where $\mathbf{n}$ is a normal at any point of the rigid boundary of the resonators. For the perturbed resonator the surface of the obstacle is assumed to be one of the rigid boundary surfaces (see Fig. 1).

It follows from Eqs. (1) and (2) and Green's theorem that

$$\int P'\nabla^2 P\, dV' - \int P\nabla^2 P'\, dV' \cong 2\omega\Delta\omega \int PP' \frac{dV'}{c^2} \tag{5}$$

or

$$\int P'\nabla P \cdot d\mathbf{A}' - \int P\nabla P' \cdot d\mathbf{A}' \cong 2\omega\Delta\omega \int PP' \frac{dV'}{c^2}. \tag{6}$$

In Eqs. (5) and (6) dV' is a volume element in the cavity of the perturbed resonator (shaded in Fig. 1) and dA' is a surface element on the rigid boundaries of the same resonator. Because $\mathbf{n} \cdot \nabla P' = 0$ for all $d\mathbf{A}'$, and $\mathbf{n} \cdot \nabla P = 0$ for all elements of area dA', except those on the surface of the obstacle, Eq. (6) reduces to

$$\int P'\nabla P \cdot d\mathbf{S} \cong 2\omega\Delta\omega \int PP' \frac{dV'}{c^2}, \tag{7}$$

where $d\mathbf{S}$ is an element of area on the surface of the obstacle in the resonator.

To evaluate the integral over dS, P' must be determined near the surface of the obstacle. Initially it is assumed that the obstacle consists of a rigid sphere of radius a, where $\omega a/c \ll 1$. With this approximation it is readily shown that Eq. (2) has the approximate solution

$$P' = P + (\nabla_0 P \cdot \mathbf{r})a^3/2r^3. \tag{8}$$

In the Eq. (8) $\nabla_0 P$ is evaluated at the center of the spherical

given by Eq. (8) satisfies the boundary equation [Eq. (4)] at the surface of the sphere. The fractional error in P' is of order of magnitude $\omega^2 a^2/c^2$.

On substituting the above expression for P' into the integral over dS [Eq. (7)], it is readily shown that

$$\int P'\nabla P \cdot dS = \int P\nabla P \cdot dS - \int_0^\pi \pi a^3 (\nabla P)^2 \times \cos^2\theta \sin\theta\, d\theta. \tag{9}$$

In the above expressions, the negative sign on the right arises because the normal to $\mathbf{dS}$ is directed toward the center of the spherical obstacle (see Fig. 1): θ is the angle between the direction of $\mathbf{a}$ and ∇P. By applying Gauss' theorem to the integral over dS [on the right-hand side of Eq. (9)], and combining Eqs. (1), (7), and (9), one obtains the following expression for $\Delta\omega$:

$$\frac{\Delta\omega}{\omega} = \left(c^2 V_s \Big/ \omega^2 \int P^2\, dV\right)\left[\left(\frac{c\, \mathrm{div}\,(\nabla P)}{\omega}\right)^2 - \frac{3(\nabla P)^2}{2}\right], \tag{10}$$

where $V_s = 4\pi a^3/3$ is the volume of the spherical obstacle. The expression $\int P^2\, dV$ in Eq. (10) replaces $\int PP'\, dV'$ of Eq. (7), since it is assumed that V_s is small compared to the total volume of the cavity resonator. Therefore, in the latter integral P' and dV' can be replaced by P and dV without introducing significant error.

The above expression for the frequency shift $\Delta\omega$ shows that it is negative, and proportional to $(\nabla P)^2$. It therefore provides no information on the direction of fluid motion. However, suppose the spherical object is made from a set of parallel disks (see Fig. 2) and assume that ∇P makes an angle ϕ with the plane of the disk. The component of ∇P parallel to the plane of the disks is therefore $(\nabla P)\cos\phi$ and the component of ∇P perpendicular to the disks is $(\nabla P)\sin\phi$. It is now assumed that these two components of ∇P contribute different amounts to $\Delta\omega/\omega$. The disks greatly impede the motion along their normal. However, the motion parallel to the plane of the disks is not greatly affected. The normal and parallel components of ∇P therefore will make different contributions to $\Delta\omega$. Since Eq. (10) shows that $\Delta\omega$ depends on $(\nabla P)^2$ for a spatially isotropic obstacle, it is reasonable to assume that the frequency shift $(\Delta\omega')$ for a spatially anisotropic obstacle may be written in the following form:

$$\Delta\omega' = [\alpha \sin^2(\phi) + \beta \cos^2(\phi)]\Delta\omega, \tag{11}$$

where $\Delta\omega$ is given by Eq. (10). The term containing α, normal to them. Conversely, $\beta \simeq 0$ and reflects the smaller frequency shifts due to the components of vibration which are parallel to the disks. It is evident from Eq. (11) that for a slotted spherical probe the resonant frequency shift depends on ϕ. The probe therefore provides information on the local direction of ∇P in the resonator. It is the purpose of the experiments described below to show that Eqs. (10) and (11) correctly describe the behavior of the perturbed resonator.

III. EXPERIMENTAL SYSTEM

The resonator is a wooden box ($0.438 \times 0.438 \times 1.18$ m) with 19.1-mm-thick plywood walls. The longest axis is arranged vertically with an 8-Ω, 10-W, 150-mm-diam speaker suspended just above the bottom by strings attached to the walls. A microphone is suspended in a rubber sling from the top endplate [Fig. 3(a)].

Two probes are used to perturb the resonant frequency [Fig. 3(b)] of the resonator. One is made from a 12.7-mm-diam aluminum rod seated in a cork bung in the neck of a 1000-ml spherical glass flask. The rod passes through a hole in the top face of the resonator. The other obstacle, which is called the "slotted spherical probe," is made from a set of 12 equally spaced circular cardboard disks whose edges are at a distance of 38.1 mm from the center of the probe. The 12.7-mm gap between the disks constitutes the slots [Fig. 3(b)].

The speaker is driven at frequencies ranging from 420 to 470 Hz, around the resonance frequency which is centered at ~445 Hz. The frequency generator is an Interstate Electronics Corporation device fitted with an external control for fine tuning the frequency.

The generator output and the microphone output are fed into a dual-trace oscilloscope and resonance is determined by adjusting the fine frequency control until the microphone output is a maximum. (Generally this method produced measurements of the resonant frequency which were consistent to 1 part in 7000.) The frequency is determined by a quartz controlled Analog Digital Research counter which measures the period of the input signal.

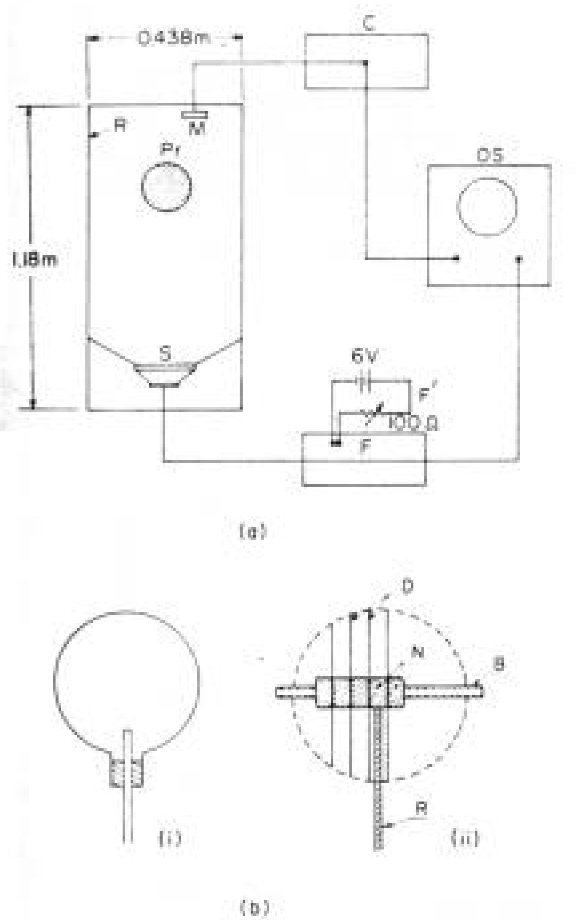
The probes are moved inside the cavity by means of the rods which pass through various holes in the walls of the resonator. The position of the probes is determined by measuring the length of the protruding support rod. For the case of the slotted sphere the orientation is determined by a protractor mounted on the resonator around the hole and a pointer mounted on the support rod.

The following measurements were performed:

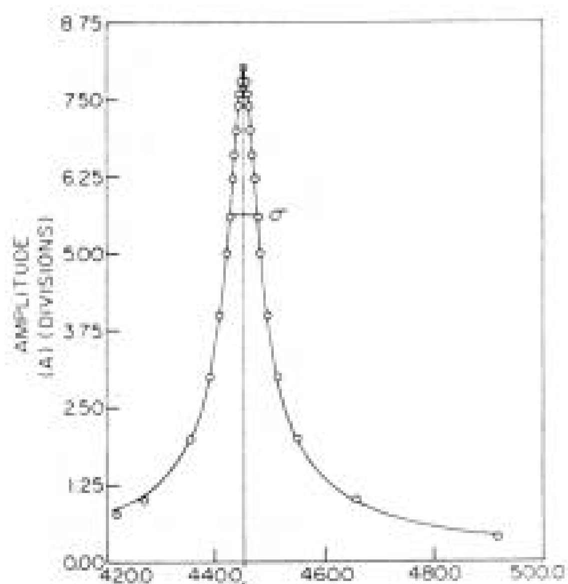
(i) the variation of resonant frequency, as a function of axial location for the spherical probe and for the slotted spherical probe with the slots perpendicular to the major axis of the resonator;

(ii) the variation of resonant frequency as a function of the angle between the plane of the slots and the major axis of the resonator (for this series of measurements the center of the probe was 581 mm from the top of the resonator cavity; and

(iii) the resonance curve for the empty resonator.



[illegible] N = spacing nuts (12.7 mm wide), B = threaded bar, and R = suspending rod.



cy = 445.41 Hz, σ = damping constant = 5.06 Hz, F = 17890 Hz^2, and reduced χ^2 = 0.00157 for 27 points.

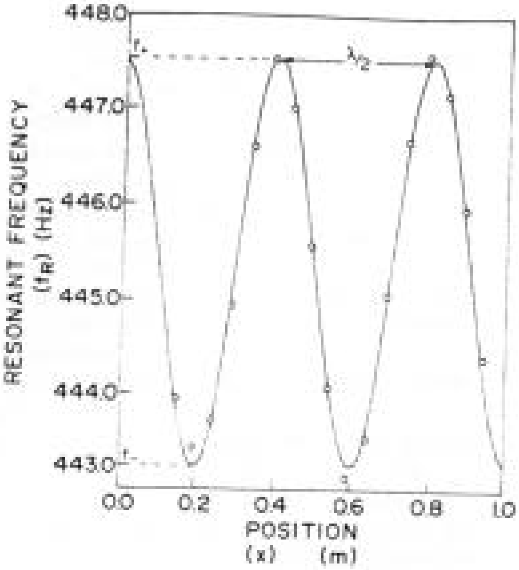


Fig. 5. Resonant frequency (f_R) as a function of axial position (x) of the spherical probe. The solid curve is $f_R = (f_+ - f_-)\cos^2[(2\pi x/\lambda) + \gamma] + f_-$. The points indicate experimental values. From the least-square fit, $f_- = 443.23$ Hz, $f_+ = 447.53$ Hz, $\lambda = 0.807$ m, $\gamma = 2.356°$, and reduced $\chi^2 = 0.0161$ with 17 points.

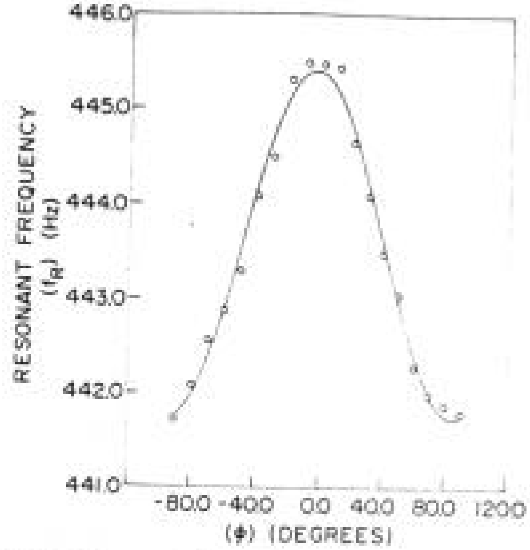
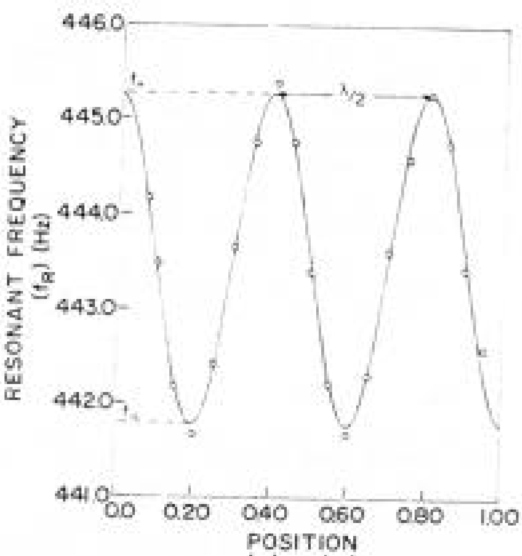


Fig. 7. Resonant frequency (f_R) as a function of orientation angle (ϕ) for the slotted spherical probe 0.581 m from the top of the resonator. The solid curve is $f_R = (f_+ - f_-)\cos^2(\phi + \phi_0) + f_-$. The points indicate experimental values. From the least-square fit, $f_- = 441.73$ Hz, $f_+ = 445.41$ Hz, $\phi_0 = 4.509°$, and reduced $\chi^2 = 0.0227$ with 19 points.

IV. RESULTS

All results were compared to theory (solid curves in Figs. 4–7) and least-square fits were used to determine the parameters which define the theoretical curves. The values of these parameters are given in the respective figure captions, together with the number of data points and the value of reduced χ^2.

Figure 4 shows a graph of microphone output versus frequency for the empty resonator. The resonator has a comparatively low quality factor Q of ~ 88, where $Q = f_R/\sigma$; f_R is the frequency at the peak in Fig. 4 and σ is the width at half-maximum power ($1/\sqrt{2}$ maximum amplitude); both values are determined by fitting the amplitude function of a forced oscillator to the experimental results.



$f_- = 441.$[illegible] Hz, $f_+ = 445.27$ Hz, $\lambda = 0.806$ m, $\gamma = 2.32°$, and reduced $\chi^2 = 0.0129$ with 19 points.

Figure 5 is a graph of f_R versus axial location x, where the 1000-ml flask is used as the obstacle; x is measured from the top of the resonator. The solid curve is the theoretical fit given by the expression

$$f_R = (f_+ - f_-)\cos^2[(2\pi x/\lambda) + \gamma] + f_-, \tag{12}$$

where f_+ and f_- are the extreme values of f_R, λ is the wavelength, and γ is a phase angle. Equation (12) follows directly from Eq. (7) if it is assumed that $(\nabla P)^2$ is proportional to $\cos^2[(2\pi x/\lambda + \gamma]$ for standing waves in the resonator. Figure 6 is a similar plot of f_R versus x with the slotted sphere mounted with the plane of its disks perpendicular to the resonator axis. Again, the graph is well described by Eq. (12).

Figure 7 shows the variation of f_R as a function of ϕ, where ϕ is the angle between the plane of the disks and the resonator axis. The center of the sphere was at $x = 0.58$ m from the top of the resonator. Figure 7 is well described by the expression

$$f_R = (f_+ - f_-)\cos^2(\phi + \phi_0) + f_-, \tag{13}$$

where ϕ_0 is a phase angle determined by the angle between ∇P and the resonator axis. The graph shows that the slotted sphere does permit one to measure the local direction of ∇P in the resonator. When f_R is maximized, the slots in the probe are parallel to ∇P.

spherical probe is useful as a device for measuring the local direction of oscillations in the resonator. One aspect of the results which call for further comment is the fact that the unperturbed resonant frequencies (i.e., the frequencies at the peaks in Figs. 4–7) appear to be variable. This occurred because no special effort was made to keep the temperature constant. Thus although temperature variations in a given run were negligible, considerable changes, often amounting to a few degrees, occurred from day to day. Since the resonant frequency of the resonator is dependent on the speed of sound in air it is evident that the temperature dependence of this latter quantity readily accounts for the observed variation of the unperturbed resonant frequency in the different experiments.

Although many standard textbooks on acoustics[1–9] consider the propagation of sound waves in pipes and the effects of obstacles on the structure of resonant cavity modes, resonant frequency shifts do not appear to be considered. The theoretical model we have used is simply a Rayleigh–Ritz perturbation calculation which is very similar to the calculation presented in Slater's book[10]; Slater considered the shift in resonant frequency produced by perturbing the shape of a microwave resonator. Morse and Feshback[11] also treat shifts in resonant frequencies caused by cavity perturbations. They give a detailed exposition of the Green's function method of doing the calculations.

The influence of perturbing obstacles on the resonant modes of an acoustic resonator has not received extensive treatment in the research literature, however. In 1974, Smith *et al.*[12] described a study of wave phenomena in an acoustic resonator. They used rectangular strips to perturb the resonant frequencies and compared their results to the detailed expressions given by Morse and Feshback[11] in terms of the mode structure for a cylindrical resonator of rectangular cross section. There has also been much interest in using acoustic modes in a space environment to levitate spheres of molten material such as one might encounter in special manufacturing processes.[13–16] Two of these studies are of particular relevance to the work presented above. In 1982, Leung *et al.*[14] used the Green's function method to calculate the resonance frequency shift of a resonator containing a rigid sphere. They point out that this problem is of importance because the levitation force on the sphere may be significantly reduced if the resonator is sufficiently detuned by the sphere. These investigators also used a disk as an obstacle and showed that the shift in resonant frequency varied sinusoidally along the axis of the resonator. However, they did not show how the shift depended on the orientation of the disk with respect to the local pressure gradient. Barmatz *et al.*,[16] in 1983, reported the results of a thorough experimental investigation of the effects of a spherical obstacle on the resonant modes of a cylindrical resonator and include the case where the sphere is no longer small compared to the wavelength of the resonant mode in the resonator.

It is evident that the main advantage of our approach to calculating the resonant frequency shift is that the result is expressed in terms of the local acoustic field, whereas the Green's function method gives an expression involving the normal mode parameters of the empty resonator. However, our approach is only valid if a is small compared to
shift caused by a spherical obstacle in a cylindrical resonator.

The cost of the experiment is quite low. Suitable speakers can be bought for less than $100 a pair and the microphone is comparable in price. The cost may also be reduced considerably by using a homemade fixed frequency driver for the speaker. In this case shifts in resonant frequency caused by the probe would be cancelled by moving a plunger in the end wall of the acoustic resonator; the measured displacement of the plunger required to maintain a constant resonant frequency would be directly proportional to the shift in resonant frequency caused by the probe. For such an experiment the microphone, frequency counter, and oscilloscope are superfluous because resonance can be determined by ear, with, however, a considerable reduction in accuracy.

It transpires that the experiment can be greatly improved at increased cost by using a microwave system. In this case the input frequency to the resonator is modulated at a frequency of a few kHz. At the same time, the perturbating obstacle is allowed to fall through the resonator. By using a measuring system such as that described by one of us elsewhere,[17] it is possible to obtain an oscillographic record of the resonant frequency of the resonator as a function of time. This record yields the shift in resonant frequency as a function of position of the probe as it falls through the resonator.

A falling obstacle cannot be used in the acoustic version of the experiment because the driver frequency of the speaker cannot be modulated rapidly enough to monitor the position of the falling probe. This conclusion follows from the fact that, for a resonator of quality factor Q, it takes roughly Q periods to establish the steady-state acoustic field. For our experiment this takes approximately 0.25 s, during which time the probe has fallen close to 0.6 m. In a typical microwave resonator the time required to establish steady conditions is typically less than a few μs. The resonant frequency can therefore be calculated by assuming that the probe is at rest each time the driving frequency satisfies the resonance condition.

For the microwave experiment it is convenient to use an axisymmetric resonator excited in a TM mode because the structure of the field is then determined by the axial electric field E_z.[10] To measure E_z one would use a probe made from a set of co-axially nested conducting circular cylindrical shells with their common axis parallel to the resonator axis, which would be vertical. The rims of each shell would be at a fixed distance from the center of the probe, which would effectively short circuit E_z but exert a minimal influence on electric fields which are perpendicular to the axis. Naturally in all experiments of this kind it is necessary to keep the frequency shifts small enough to avoid resonant coupling to modes whose geometry is different from the one which is under investigation.

A further advantage of a microwave version of the experiment is the higher spatial resolution which can be achieved. It is readily shown from the theoretical model [Eq. (10)] that the order of magnitude of the observed frequency shift is $f_R L_P^3/L_R^3$, where L_P is a typical linear dimension of the probe and L_R is a typical characteristic length for the resonator. To be measurable, this shift must
$(100Q)^{1/3}$. Since microwave resonators with a Q exceeding

10 000 are not hard to achieve, a typical probe diameter for a microwave experiment will be something like 1% of the characteristic scale length for the microwave resonator. For the acoustic resonator described above, $Q = 100$, which means that L_p/L_R must exceed 4% or 5% in acoustic experiments. Hence microwave resonators are also to be preferred because of the improved spatial resolution they afford.

Perhaps the greatest advantage of the microwave version of the experiment is that it would be possible to map the complete field distribution in the resonator and that this task could be accomplished rapidly, an objective not attainable with the acoustic version described above.

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