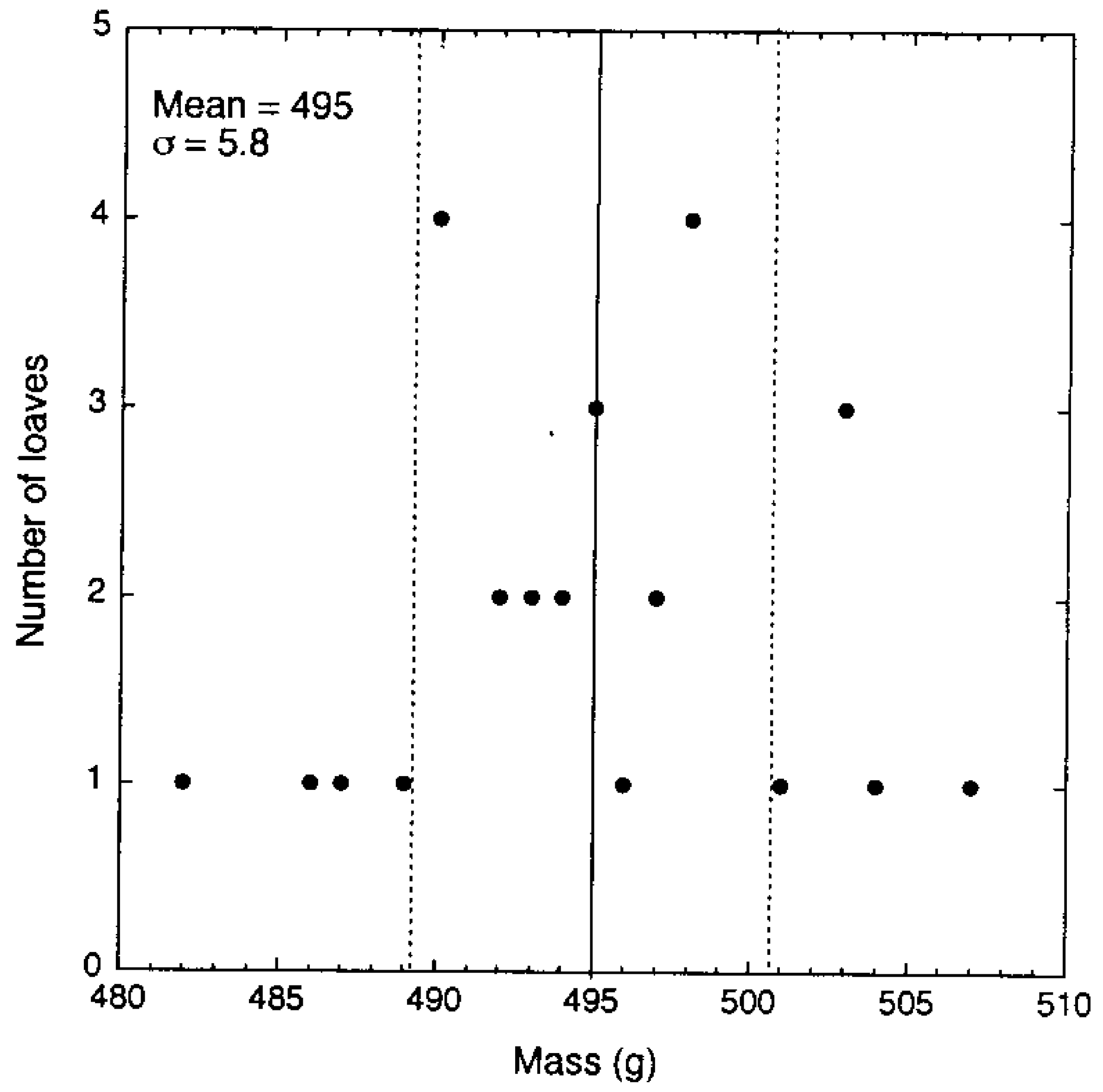
Phys 150 HW3 solutions ①

$\boxed{C3.1}$ (a) You issue the bakery a warning because the average mass of a loaf of bread is under 500g, the state mass being sold.

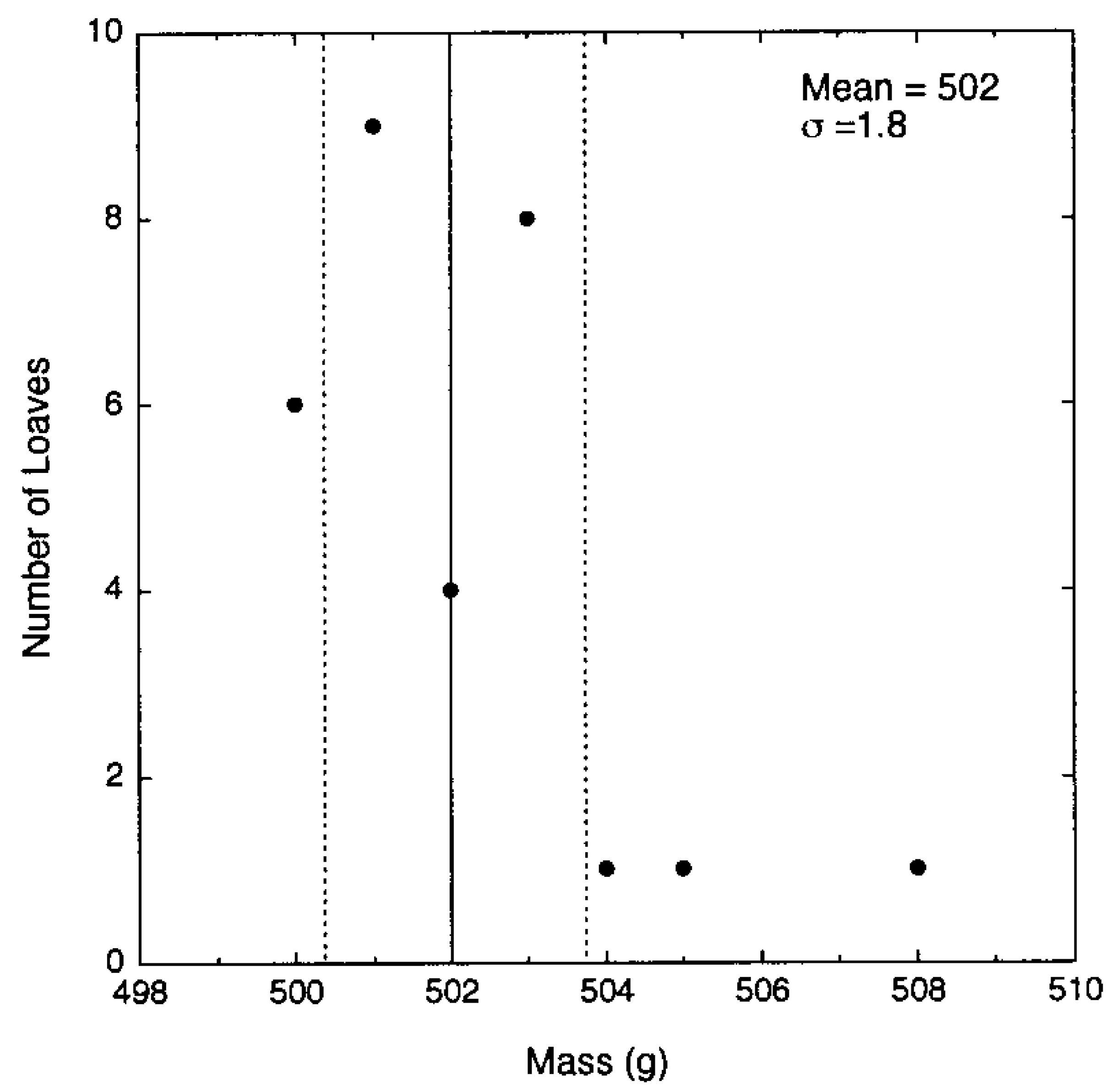
Initial inspection →



$$\langle m \rangle = \sum_{i=1}^{N} \frac{m_i}{N}$$

For the thirty loaves $\boxed{\langle m \rangle \sim 495\ g}$

(b) second inspection



The bakery sold you loaves of bread with an average mass of ~502g during your return visit.

Your boss wants you to shut the shop down even though the average loaf that you bought has a mass of more than the stated 500g (and all of the loaves had a mass of at least 500g).

Standard deviation: $\sigma = \sqrt{\langle (m_i - \langle m_i \rangle)^2 \rangle}$

She could have noticed that the standard deviation of the distributions in the first and second inspections differed. It is unlikely that the bakery was able to change how reproducibly they can make a loaf of bread of a given mass, when they increased the average mass of a loaf of bread by a few grams. The narrower distribution of masses in the second inspection suggests that these loaves may have been specially selected from the larger distribution just for you.

Your boss may have also noticed that the second distribution appears truncated – there are more infrequently occuring masses above the mean than below the mean. In the initial, surprise, inspection the distribution appears more symmetric (see plot in (a)). This also suggests that these loaves were specially selected for you (and that the true mean might be less than 500 g).

$$\sigma = \sqrt{\frac{1}{30} \sum_{i=1}^{30} (m_i - \langle m_i \rangle)^2} = \sqrt{\frac{1}{30} (m_1 - 502)^2 + (m_2 - 502)^2 + \ldots + (m_{30} - 502)^2}$$

3.4 (a) Can a single photon deliver enough energy to a volume the size of a cell nucleous ~~such~~ that the chromosomes inside the nucleous ~~break~~ apart due to increased thermal motion?

Suppose that a cell nucleus is a sphere of radius $1\,\mu m$ with the density of $H_2O = \frac{1g}{cm^3}$

Then the volume, $V = \frac{4\pi r^3}{3} \approx 4\,\mu m^3$

We find in Appendix B that $1.6 \cdot 10^{-19}\,J = 1\,eV$

And we recall that it takes $1\,cal$ to raise $1\,g$ of $H_2O \sim 1\,K$

$$\Delta T = \frac{10\,eV}{4\,\mu m^3} \cdot \frac{1.6 \cdot 10^{-19}\,J}{eV} \cdot \frac{0.24\,cal}{J} \cdot \frac{1\,cm^3}{1g} \cdot \frac{10^{12}\,\mu m^3}{1\,cm^3} \cdot \frac{1g\ 1K}{1\,cal} \sim 0.1 \cdot 10^{-6}\,K = \boxed{0.1\,\mu K}$$

This is far too small of a temperature change to cause damage through increased thermal motion.

(b) $\Delta T = 70°C = 70K$ Same calculation as in (a) only this time we know the temperature change and are looking for what volume would undergo this temperature change with the addition of a single photon's energy.

$$V = \frac{10\,eV}{70\,K} \cdot \frac{1K \cdot 1g}{1\,cal} \cdot \frac{cm^3}{1g} \cdot \frac{1.6 \cdot 10^{-19}\,J}{eV} \cdot \frac{0.24\,cal}{J} = 5.5 \cdot 10^{-21}\,cm^3 \sim \boxed{6\,nm^3}$$

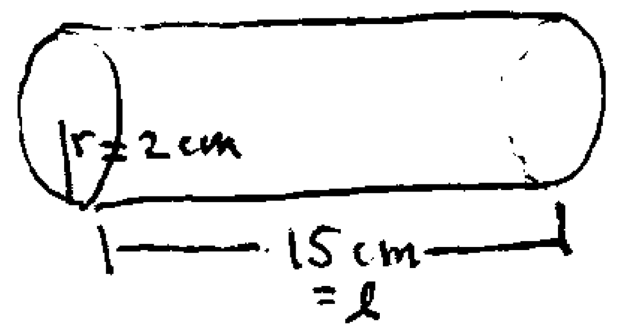
So we could estimate the size of a gene to be a few cubic nanometers if this proposal is correct.

C3.6 What is the probability of getting at least one five if you roll three fair dice?

Easier to calculate the probability of never rolling a five and subtract from 1

$$1 - \left(\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}\right) = \boxed{0.42}$$ is the probability of rolling at least one five

C3.8 (a) Crystal about the size of a tube of toothpaste:



$V = \pi r^2 l$

$= \pi \cdot 4 \cdot 15\,cm^3 \sim 200\,cm^3$

Atoms spaced 0.4 nm apart; information stored in defects in the crystal lattice. (1 defect = 1 bit)

How many bits of information can the crystal contain?

The number of bits of information is just the number of defects. Need to assume a certain defect density to solve this. To maintain a recognizable crystal lattice, let's say that we need ~4 repeat units in each direction, so $\sim 5^3 \sim 100$ atoms. Then we can have $\sim \frac{1\ \text{defect}}{100\ \text{atoms}}$

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C3.8 (a) continued

The number of defects is just $\left(\# \text{atoms} = \frac{V_{crystal}}{V_{atom}} \right) \cdot \frac{\# \text{defects}}{\text{atom}}$

$$\frac{200 \text{ cm}^3}{(0.4 \cdot 10^{-7} \text{cm})^3} \cdot \frac{1}{100} \sim 10 \cdot 10^{21} \sim 10^{22} \text{ defects}$$

So about $\boxed{10^{22} \text{ bits}}$ of information can be stored in the crystal

(b) 28 known galaxies, encyclopedia $\sim 38 \cdot 10^9$ bits

The encyclopedias per known galaxy stored on the crystal is just

$$\# \text{bits} \cdot \frac{\# \text{ encyclopedias}}{\# \text{ bits}} \cdot \frac{1}{\# \text{ galaxies}}$$

$$\sim 10^{22} \text{ bits} \cdot \frac{1 \text{ encyclopedia}}{4 \cdot 10^{10} \text{ bits}} \cdot \frac{1}{3 \cdot 10 \text{ galaxies}} \sim \frac{10^{22}}{10 \cdot 10^{11}} \sim \boxed{10^{10} \frac{\text{encyclopedias}}{\text{galaxy}}}$$

C3.9 (a) See next page.
Note x_0 is the true mean, a theoretical value that is not the result of a series of measurements.

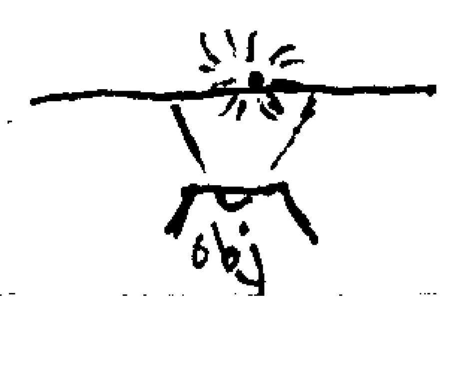
(b) single fluorophore emits $\sim 10^6$ photons before photobleaches

We can estimate a limit on the accuracy with which we can measure the

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$ where σ_x is approximate wavelength of light (we will use 500nm) and N is the number of photons

Then $\sigma_{\bar{x}} = \frac{500\text{nm}}{\sqrt{10^6}} = \frac{500\text{nm}}{10^3} = \boxed{0.5\text{nm}}$

In practice, this limit is not achieved because all of the photons emitted are not detected.



First, microscope objectives only collect light from one side of the sample
Second, only a portion of the photons emitted on the sample side facing the objective are incident on the lens of the objective, and so are collected.

C3.9 (a)

The standard deviation, σ_x, of a measurement, x_i, from it's central value, x_o, is given by

$$\sqrt{\left\langle (x_i - x_o)^2 \right\rangle}. \tag{1}$$

The standard deviation, $\sigma_{\bar{x}}$, of the mean of multiple measurements, $\bar{x}$ from the central value, x_o,is given by

$$\sqrt{\left\langle (\bar{x} - x_o)^2 \right\rangle}. \tag{2}$$

To see the difference between these two standard deviations, it helps to make the averages implied by the angle brackets and the over-bar explicit.

For the standard deviation of the individual measurements,

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - x_o)^2}{N}}. \tag{3}$$

For the standard deviation of the mean of multiple measurements,

$$\sigma_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{N} \left(\frac{\sum_{i=1}^{N} x_i}{N} - x_o \right)^2}{N}} \tag{4}$$

Focusing on the term inside the parentheses, note that

$$\left(\frac{\sum_{i=1}^{N} x_i}{N} - x_o \right)^2 = \left(\frac{\left(\sum_{i=1}^{N} x_i \right) - N x_o}{N} \right)^2 = \left(\frac{\sum_{i=1}^{N} (x_i - x_o)}{N} \right)^2 = \frac{\sum_{i=1}^{N} (x_i - x_o)^2 + \sum_{i \neq j}^{N} 2 (x_i - x_o)(x_j - x_o)}{N^2} \tag{5}$$

Because the i^{th} measurement is independent of the j^{th} measurement, the argument of the sum in second term in the numerator is negative as often as it is positive and it sums to zero. Meaning we can rewrite

$$\sigma_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{N} \left(\frac{\sum_{i=1}^{N} (x_i - x_o)^2}{N^2} \right)}{N}} = \sqrt{\frac{\sum_{i=1}^{N} \sum_{i=1}^{N} \frac{(x_i - x_o)^2}{N^2}}{N}} = \sqrt{\frac{\sum_{i=1}^{N} \frac{N (x_i - x_o)^2}{N^2}}{N}} = \sqrt{\frac{\sum_{i=1}^{N} \frac{(x_i - x_o)^2}{N}}{N}} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - x_o)^2}{N^2}} \tag{6}$$

Comparing

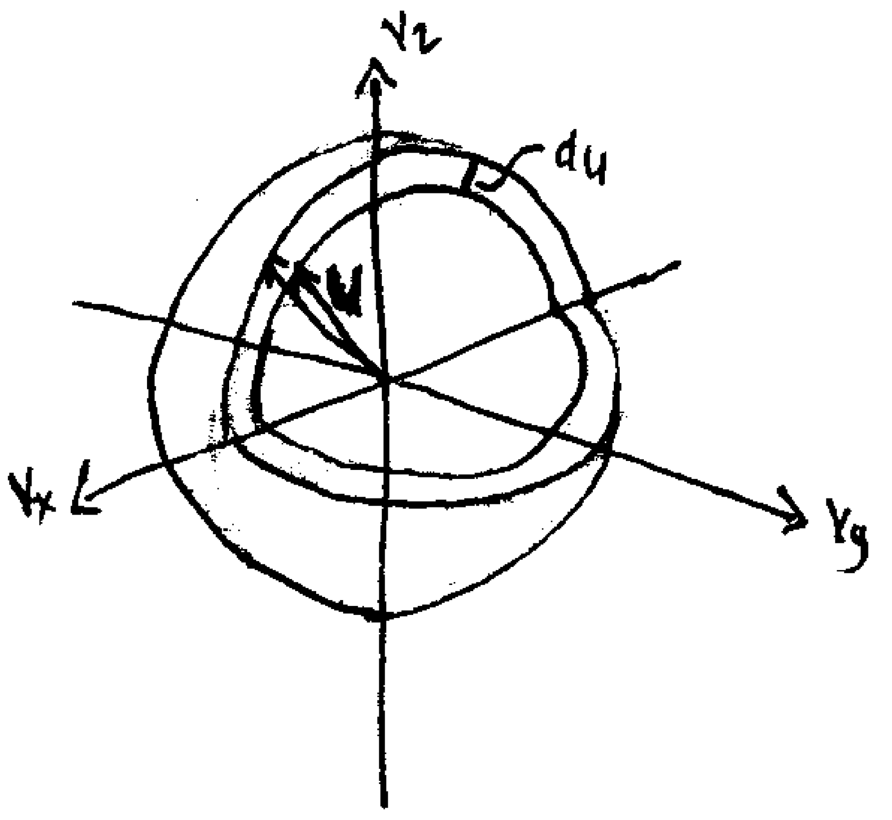
$$\sigma_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - x_o)^2}{N^2}} \quad \text{and} \quad \sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - x_o)^2}{N}} \tag{7}$$

We see that

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} \tag{8}$$

This makes sense: the larger the number of measurements, the less the average of those measurements will deviate from the true mean of their distribution.

3F YT (a) Find the probability $P(u)du$ that an arrow ends up in the volume $u + du$ from the target which we will center at the origin, for convenience.



$$u = |\vec{v}| = \sqrt{x^2+y^2+z^2}$$

Each component of $\vec{v}$ is an independent, random variable distributed as a Gaussian with variance σ^2

So $P_x = (2\pi\sigma^2)^{-1/2}\, e^{-x^2/2\sigma^2}$.

$$P_{xyz}(x,y,z) = P_x(x)dx \cdot P_y(y)\,dy \cdot P_z(z)dz$$

$$= (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} dx \cdot (2\pi\sigma^2)^{-1/2} e^{-y^2/2\sigma^2} dy \cdot (2\pi\sigma^2)^{-1/2} e^{-z^2/2\sigma^2} dz$$

$$= (2\pi\sigma^2)^{-3/2}\, e^{-(x^2+y^2+z^2)/2\sigma^2}\, dxdydz$$

$$d^3v = dxdydz = 4\pi u^2 du$$

$$P(u)\,du = (2\pi\sigma^2)^{-3/2}\, e^{-u^2/2\sigma^2}\, 4\pi u^2 du$$

(b) Graph on computer

3L The most probable value of the speed u is when $\frac{dP(u)}{du} = 0$

$$\frac{d}{du}P(u) = \frac{4\pi}{(2\pi\sigma^2)^{3/2}}\left[2u\,e^{-u^2/2\sigma^2} + u^2 e^{-u^2/2\sigma^2} \cdot \left(-\frac{2u}{2\sigma^2}\right)\right]$$

$$= \frac{4\pi}{(2\pi\sigma^2)^{3/2}}\left[2u\,e^{-u^2/2\sigma^2} - \frac{u^3}{\sigma^2}\, e^{-u^2/2\sigma^2}\right]$$

$\frac{d}{du}P(u) = 0$ when $2u\,e^{-u^2/2\sigma^2} = \frac{u^3}{\sigma^2}\, e^{-u^2/2\sigma^2} \Rightarrow u^2 = 2\sigma^2 \Rightarrow \boxed{u = \sqrt{2}\,\sigma}$

The mean speed $\langle u \rangle$ is defined as $\int du\, u\, P(u)$

$$\langle u \rangle = \frac{4\pi}{(2\pi\sigma^2)^{3/2}} \int_0^\infty u^3 e^{-u^2/2\sigma^2}\, du$$

To find the mean speed, we evaluate this integral.

3L continued

$$\langle u \rangle = \frac{4\pi}{(2\pi\sigma^2)^{3/2}} \int_0^\infty u^3 e^{-u^2/2\sigma^2} du$$

First we make a substitution, Letting $y = -\frac{u^2}{2\sigma^2}$, $dy = -\frac{u}{\sigma^2} du$

$$\Rightarrow \frac{4\pi}{(2\pi\sigma^2)^{3/2}} (2\sigma^2)\sigma^2 \int_0^\infty y e^{y} dy$$

Then we integrate by parts: $\int k\,dl = kl - \int l\,dk$

Let $k = y$ $\quad dl = e^y dy$

$dk = dy$ $\quad l = e^y$

$$= y e^y \Big|_0^\infty - \int_0^\infty e^y dy$$

$$= \Big[y e^y - e^y \Big]_0^\infty$$

Returning to the original variable, u and the constants

$$= \frac{4\pi}{(2\pi\sigma^2)^{3/2}} 2\sigma^4 \underbrace{\left(-\frac{u^2}{2\sigma^2} - 1 \right) e^{-u^2/2\sigma^2}}_{} \Bigg|_0^\infty$$

This term goes to zero in the limit as $u \to \infty$
one (negative) $u \to 0$

$$= \frac{4\pi}{(2\pi\sigma^2)^{3/2}} 2\sigma^4$$

$$\langle u \rangle = \boxed{\frac{2}{\sqrt{\pi}} \sigma}$$