

Phys 150 HW 5 solutions

Nelson, Problem 5.2, pg 190 (4 pts)

To find the constant pre-factor, recognize that the integral of the concentration throughout the volume has to be equal to the total number of particles in the volume

$$\int c(z)dV = A \int_0^h c(z)dz = N. \quad (1)$$

Substituting in from equation 5.1

$$A \int_0^h c(0)e^{-m_{\text{net}}gz/k_B T} dz = N \quad (2)$$

$$-c(0)A \frac{k_B T}{m_{\text{net}}g} e^{-m_{\text{net}}gz/k_B T} \Big|_{z=0}^{z=h} = N \quad (3)$$

$$-c(0)A \frac{k_B T}{m_{\text{net}}g} \left(e^{-m_{\text{net}}gh/k_B T} - 1 \right) = N \quad (4)$$

$$c(0) \left(1 - e^{-m_{\text{net}}gh/k_B T} \right) = \frac{Nm_{\text{net}}g}{Ak_B T} \quad (5)$$

$$c(0) = \frac{Nm_{\text{net}}g}{Ak_B T} \left(1 - e^{-m_{\text{net}}gh/k_B T} \right)^{-1} \quad (6)$$

Therefore,

$$c(z) = \frac{Nm_{\text{net}}g}{Ak_B T} \frac{e^{-m_{\text{net}}gz/k_B T}}{1 - e^{-m_{\text{net}}gh/k_B T}} \quad (7)$$

Nelson, Problem 5.3, pg 190 (8 pts)

a) The sedimentation coefficient s is defined as the ratio between the effective mass and the drag coefficient,

$$s \equiv m_{\text{net}}/\zeta. \quad (8)$$

When the drift velocity was due to gravity, $F = v_{\text{drift}}\zeta = m_{\text{net}}g$, and $s = v_{\text{drift}}/g$. In the centrifuge, the drift velocity (in the rotating frame of the particle) is due to the centrifugal force $F = v_{\text{drift}}\zeta = m_{\text{net}}\omega^2 r$. The analogous relationship between drift speed and s is therefore, $s = v_{\text{drift}}/(\omega^2 r)$ or $v_{\text{drift}} = \omega^2 r s$.

b) Recall that Fick's law gives the flux due to a concentration gradient:

$$j_x = -D \frac{dc(x)}{dx}. \quad (9)$$

In the rotating frame of the centrifuge, the x-coordinate is the radial position. At the far ends of the tube

$$j_r = -D \frac{dc(r)}{dr} + v_{\text{drift}}c(r) = 0, \quad (10)$$

where $r = r_1$ or $r = r_2 = r_1 + \ell$. Therefore

$$v_{\text{drift}}c(r_1) = D \frac{dc}{dr} \Big|_{r=r_1} \text{ and } v_{\text{drift}}c(r_2) = D \frac{dc}{dr} \Big|_{r=r_2} \quad (11)$$

c) Substituting in for v_{drift} in terms of m_{net} , solving for m_{net} and invoking the Einstein relation $D\zeta = k_B T$

$$\frac{m_{\text{net}}\omega^2 r_1}{\zeta} c(r_1) = D \frac{dc}{dr} \Big|_{r=r_1} \quad (12)$$

$$m_{\text{net}} = \frac{D\zeta}{\omega^2 r_1 c(r_1)} \frac{dc}{dr} \Big|_{r=r_1} = \frac{k_B T}{\omega^2 r_1 c(r_1)} \frac{dc}{dr} \Big|_{r=r_1}. \quad (13)$$

where all the terms on the right hand side of the final equation are measurable.

Nelson, Problem 5.4, pg 191 (4 pts)

a) The Stokes drag force is $\vec{F} = -6\pi\eta R\vec{v}$. Newton's Law of motion states $\vec{F} = m\vec{a}$. Combining and solving for v

$$m\dot{v} = -6\pi\eta Rv \quad (14)$$

$$\dot{v}/v = -6\pi\eta R/m \quad (15)$$

$$\ln v(t) = -6\pi\eta Rt/m + \text{const} \quad (16)$$

$$v(t) = v_0 e^{-6\pi\eta Rt/m} \quad (17)$$

Knowing $v(t)$, we can solve for $x(t)$ by integrating

$$\Delta x = \int_0^\infty v(t) dt \quad (18)$$

$$= v_0 \int_0^\infty e^{-6\pi\eta Rt/m} dt \quad (19)$$

$$= -v_0 \frac{m}{6\pi\eta R} e^{-6\pi\eta Rt/m} \Big|_0^\infty \quad (20)$$

$$= \frac{mv_0}{6\pi\eta R} \quad (21)$$

Given $R = 10^{-6}\text{m}$, $\eta = 9 \cdot 10^{-4}\text{Pa s}$, $v_0 = 10^{-6}\text{m s}^{-1}$, $m = \frac{4}{3}\pi R^3 \Delta\rho$ and assuming $\Delta\rho = (\rho_{\text{bacterium}} - \rho_{\text{water}}) = 0.25$, we find $\Delta x \approx 10^{-12}\text{m}$, which is much less than the size of an atom!

b) This assumption is very well justified in light of (a). The particle loses any drift velocity almost immediately after the force ceases.

Nelson, Problem 5.5, pg 191 (6 pts)

a) Referring to the Hagen-Poiseuille relation for laminar pipe flow (given on Nelson, page 181)

$$\frac{p}{L} = \frac{8Q\eta}{\pi r^4} = \frac{8 \cdot (500 \cdot 10^{-6}\text{m}^3\text{s}^{-1}) \cdot (9 \cdot 10^{-4}\text{Pa s})}{\pi \cdot (0.125 \cdot 10^{-1}\text{m})^4} = \frac{8 \cdot 5 \cdot 9 \cdot 10^{-8} \text{ Pa}}{\pi \cdot (1/8)^4 \cdot 10^{-4} \text{ m}} = 47 \text{ Pa m}^{-1} \quad (22)$$

Across a 10 cm section of aorta, the pressure drop is therefore only about 5 Pa, which is 0.01% of atmospheric pressure.

b) The power expended is the work done per unit time. The work done can be computed from the pressure drop (force per unit area) times the flow rate (distance \cdot area per unit time).

$$pQ = 47 \frac{\text{Pa}}{\text{m}} \cdot 0.1 \text{ m} \cdot 500 \cdot 10^{-6} \frac{\text{m}^3}{\text{s}} = 2350 \cdot 10^{-6} \frac{\text{N m}}{\text{s}} = 2.4 \text{ mW} \quad (23)$$

This is more than four orders of magnitude less than the basal metabolic rate, which means that very little of the energy required to sustain life is needed to maintain blood flow.

c) Referencing Equation 5.17 on page 181, the sketch of $v(r) = (R^2 - r^2)p/(4L\eta)$ should be parabolic. To find $v(r=0) = R^2 p/(4L\eta)$ note that

$$Q = \frac{\pi R^4 p}{8L\eta} \rightarrow v(0) = \frac{R^2 p}{4L\eta} = \frac{2Q}{\pi R^2} = \frac{1000 \cdot 10^{-6}\text{m}^3\text{s}^{-1}}{\pi(1.25 \cdot 10^{-2}\text{m})^2} = 20 \text{ m/s}. \quad (24)$$

Nelson, Your Turn 5C, pg 161 (5 pts)

a) Combining the relation for viscous drag $F = \zeta v$ with Stokes formula $\zeta = 6\pi\eta R$, gives

$$F = 6\pi\eta Rv \rightarrow \eta = \frac{F}{6\pi Rv} = \frac{\text{Force}}{\mathbb{L} \cdot \mathbb{L}\mathbb{T}^{-1}} = \frac{\text{Force}}{\mathbb{L}^2 \cdot \mathbb{T}^{-1}} = \frac{\text{Pressure}}{\mathbb{T}^{-1}} = \text{Pressure} \cdot \text{time} \quad (25)$$

b) The drift velocity due to the force of gravity acting on a fat droplet of radius R is

$$v_{\text{drift}} = \frac{m_{\text{net}}g}{6\pi\eta R} = \frac{\Delta\rho_m 4\pi R^3 g}{3 \cdot 6\pi\eta R} = \frac{2\Delta\rho_m R^2 g}{9\eta} \quad (26)$$

The mass density of butterfat is $\approx 0.9 \text{ g cm}^{-3}$, so $\Delta\rho_m \approx 0.1 \text{ g cm}^{-3} = 100 \text{ kg m}^{-3}$. The viscosity of milk is nearly the same as that of water, $\eta = 9 \cdot 10^{-4} \text{ Pa s}$. Plugging in these values and calculating the drift velocity for a droplet with $R = 0.5 \cdot 10^{-6} \text{ m}$ gives

$$v_{\text{drift}} = \frac{2 \cdot 100 \cdot (0.5 \cdot 10^{-6})^2 \cdot 10}{9 \cdot 9 \cdot 10^{-4}} = \frac{5 \cdot 10^{-10}}{81 \cdot 10^{-4}} \approx 6 \cdot 10^{-8} \text{ m/s} \quad (27)$$

Seeing as how a typical milk bottle is $\approx 0.25 \text{ m}$ tall, it would take $4 \cdot 10^6 \text{ s} \approx 45$ days for such small fat droplets to accumulate at the top of the milk container. On the other hand, if the droplets were 5 times larger, the drift velocity would be 25 times faster and separation would occur in only one or two days!

Nelson, Problem 5.11, pg 584 (2 pts) In striving to maintain a constant total flow rate as R decreases, the blood pressure, p rises in proportion to R^{-4} , which in turn causes the flow velocity v to increase in proportion to R^{-2} . The Reynold's number of the flow, $\mathcal{R} = vR\rho_m/\eta$, therefore increases in proportion to $1/R$ and, when it becomes $\mathcal{R} \gg 1$, the flow switches from laminar to turbulent.

Nelson, Your Turn 5F, pg 171 (4 pts) If $c_1(x, t)$ is a solution to the diffusion equation $\dot{c}_1(x, t) = Dc_1''(x, t)$ then $c_2 \equiv c_1(x, -t)$ is not because, although $c_2'' = c_1''$, $\dot{c}_2 = -\dot{c}_1 \neq \dot{c}_1$. In other words, the diffusion equation is not invariant under time-reversal, which is a mathematical way of saying that diffusion is an irreversible process.

By contrast, a function $c_3(x, t) \equiv c_1(-x, t)$ is a solution of the diffusion equation because, although $c_3' = -c_1'$, $c_3'' = c_1''$. In other words, diffusion is a spatially invariant process.

Nelson, Problem 5.12, pg 584 (4 pts)

a) The rate at which the apparatus does work on the fiber is given by the time derivative of the product of the force that the apparatus exerts on the fiber (the opposite of the force given in the problem statement) times the change in the fiber's length.

$$\text{Power} = \frac{d\text{Work}}{dt} = \frac{d}{dt} (\vec{f}(t) \cdot d\vec{x}(t)) \quad (28)$$

$$= \frac{d}{dt} ((f_0 + B \sin(\omega t + \delta))(A\omega \cos(\omega t))) \quad (29)$$

$$= \frac{d}{dt} (f_0 A\omega \cos(\omega t) + AB\omega \sin(\omega t + \delta) \cos(\omega t)) \quad (30)$$

$$= -f_0 A\omega^2 \sin(\omega t) + AB\omega^2 (\cos(\omega t + \delta) \cos(\omega t) - \sin(\omega t + \delta) \sin(\omega t)) \quad (31)$$

$$= AB\omega^2 \cos(2\omega t + \delta) - f_0 A\omega \cos(\omega t) \quad (32)$$

which averages to zero over one full cycle because every term is periodic with frequency of either ω or 2ω .

b) Number 1 is the living muscle fiber and Number 2 is the dead one. The key thing to note is that Number 1 displays negative values of the phase shift, which means that sometimes "the displacement leads the force" rather than lagging it. To lead the force requires putting work into the oscillating system. This can only be done by a fiber that can add mechanical energy to the system - in this case a living muscle fiber that transforms chemical energy into mechanical energy.