PHYS 21: Assignment 2 Solutions

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Prof. Oh

Katharine Hyatt

6.7

We place one of the vertices at the origin, and then place the other two so that the side across from the origin-vertex is vertical. It's located at $x = \frac{\sqrt{3}L}{2}$. Then:

$$\begin{aligned} dm =& \frac{M}{A}\frac{2x}{\sqrt{3}}dx \\ =& \frac{4M}{\sqrt{3}L^2}\frac{2x}{\sqrt{3}}dx \\ =& \frac{8M}{3L^2}xdx \end{aligned}$$

We want to find the moment of inertia about a vertex, not about the centre of mass, so we need the Parallel Axis Theorem:

$$\begin{aligned} dI =& dI_{CoM} + x^2 dm \\ =& \frac{1}{12}\frac{4x^2}{3}dm + x^2 dm \\ =& \frac{10}{9}x^2 dm \\ I =& \frac{10}{9}\int x^2 dm \\ =& \frac{10}{9}\frac{8M}{3L^2}\int_0^{\sqrt{3}L/2} x^3 dx \\ =& \frac{5}{12}ML^2 \end{aligned}$$

Our units are the correct ones for moments of inertia.

6.9

First, we need to sum up all the forces in this problem. Since no part of the system is accelerating vertically, we can say that:

$$\sum F_y = 0$$

Friction in this problem is entirely horizontal, and the normal forces are entirely vertical. We can write:

$$\begin{aligned} \sum F_y =& N_1 + N_2 - Mg \\ M_g =& N_1 + N_2 \end{aligned}$$

The sum of the horizontal forces may not be zero (if it were, the bar would never roll, since it starts out at rest). We see that f_1 and f_2 are in opposite directions so:

$$\begin{aligned} \sum F_x =& f_2 - f_1 \\ =& \mu(N_1 - N_2) \end{aligned}$$

We need to get expressions for N_1 and N_2 now. We can do this by considering the sum of the torques. The frictions will exert no torque since they lie parallel to the moment arm from the centre of mass of the bar. Gravity exerts no toque because the moment arm would have zero length.

We see that the system isn't rotating, so:

$$\begin{aligned} 0 =& \sum \tau \\ =& N_1(\ell + x) - N_2(\ell - x) \\ N_1(\ell + x) =& N_2(\ell - x) \end{aligned}$$

Since N_1, N_2 are parallel, their torques must have opposite sign. In this case we have chosen the positive x direction to be towards the left.

$$\begin{aligned} N_1 =& N_2 \frac{\ell - x}{\ell + x} \\ Mg =& N_2 \left(1 + \frac{\ell - x}{\ell + x}\right) \\ N_2 =& \frac{Mg(\ell + x)}{\ell + x + \ell - x} \\ =& \frac{Mg}{2}\left(1 + \frac{x}{\ell}\right) \\ N_1 =& \frac{Mg}{2}\left(1 + \frac{x}{\ell}\right)\frac{\ell - x}{\ell + x} \\ =& \frac{Mg}{2}\left(\frac{\ell^2 - \ell x + x\ell - x^2}{\ell(\ell + x)}\right) \\ =& \frac{Mg}{2}\left(\frac{\ell^2 - x^2}{\ell(\ell + x)}\right) \\ =& \frac{Mg}{2}\left(\frac{\ell - x}{\ell}\right) \\ =& \frac{Mg}{2}\left(1 - \frac{x}{\ell}\right) \end{aligned}$$

Now that we have the normal forces, we can find the horizontal force:

$$\begin{aligned} \Rightarrow \sum F_x =& \frac{Mg\mu}{2}\left(1 - \frac{x}{\ell} - 1 - \frac{x}{\ell}\right) \\ =& -\frac{Mg\mu}{\ell}x \\ M\ddot{x} =& -\frac{Mg\mu}{\ell}x \\ 0 =& M\ddot{x} + \frac{Mg\mu}{\ell}x \\ =& \ddot{x} + \frac{g\mu}{\ell}x \end{aligned}$$

We see that this is exactly the differential equation describing simple harmonic motion, since g, μ, and ℓ are all positive. Then we define $\omega = \sqrt{\frac{g\mu}{\ell}}$ so that $x = x_0 \cos \omega t$ - there is only a cosine term because $\dot{x}(0) = 0$ - the system is initially at rest.

6.10

In this case, the sum of the forces in every direction should be zero since the system isn't accelerating in any direction. The sum of the torques also needs to be zero if we would like to maintain the angular velocity of the roller. Let's look at torque first. Since gravity acts at the centre of mass, it exerts no torque. The normal forces also don't exert any torque since they point directly towards the centre of mass. Then:

$$\begin{aligned} 0 &= \sum \tau \\ &= \tau_{applied} - (f_1 + f_2)R \\ \tau_{applied} &= (f_1 + f_2)R \\ &= \mu(N_1 + N_2)R \end{aligned}$$

Now to find N_1 and N_2, we will balance the forces in this problem:

$$\begin{aligned} \vec{0} &= \sum \vec{F} \\ 0 &= \sum F_y \\ &= -Mg + N_1 \cos\frac{\pi}{4} + N_2 \cos\frac{\pi}{4} + f_1 \cos\frac{\pi}{4} - f_2 \cos\frac{\pi}{4} \\ Mg\sqrt{2} &= N_1(1+\mu) + N_2(1-\mu) \\ 0 &= \sum F_x \\ &= N_1 \sin\frac{\pi}{4} - N_2 \sin\frac{\pi}{4} - f_1 \sin\frac{\pi}{4} - f_2 \sin\frac{\pi}{4} \\ &= N_1 - N_2 - \mu N_1 - \mu N_2 \\ N_1(1-\mu) &= N_2(1+\mu) \\ N_1 &= N_2 \frac{1+\mu}{1-\mu} \\ Mg\sqrt{2} &= N_2 \frac{(1+\mu)^2}{1-\mu} + N_2(1-\mu) \\ &= N_2 \frac{(1+\mu)^2 + (1-\mu)^2}{1-\mu} \\ N_2 &= \frac{\sqrt{2}Mg(1-\mu)}{1 + 2\mu + \mu^2 + 1 - 2\mu + \mu^2} \\ N_2 &= \frac{Mg(1-\mu)}{\sqrt{2}(1+\mu^2)} \\ N_1 &= \frac{Mg(1+\mu)}{\sqrt{2}(1+\mu^2)} \end{aligned}$$

Now we can plug these normal forces back in to find the torque we need to apply:

$$\begin{aligned} \tau_{applied} &= \frac{MgR\mu}{\sqrt{2}(1+\mu^2)}(1 + \mu + 1 - \mu) \\ &= \frac{\sqrt{2}MgR\mu}{1+\mu^2} \end{aligned}$$

6.11

We know that at a time t_0, a length L of tape has been unwound. We also know that the drum is rotated by a constant force F applied a distance R from its centre. At t_0, the wheel rotates with speed ω_0. So:

$$\begin{aligned}\vec{\tau} =& \vec{r} \times \vec{F}\\ \tau =& Fr\end{aligned}$$

And we also know that:

$$\begin{aligned}\tau =& I_0 \frac{d\omega}{dt}\\ FR =& I_0 \frac{d\omega}{dt}\\ \frac{FR}{I_0} =& \frac{d\omega}{dt}\end{aligned}$$

The left side of this equation is constant, so we can see that ω varies linearly with time!

$$\begin{aligned}\omega(t) =& \frac{FR}{I_0} t\\ \frac{d\theta}{dt} =& \frac{FR}{I_0} t\\ \int_{\theta(0)}^{\theta(t)} d\theta =& \int_0^t \frac{FR}{I_0} t dt\\ \theta(t) - \theta(0) =& \frac{FR}{2I_0} t^2\\ \theta(t) =& \frac{FR}{2I_0} t^2\end{aligned}$$

We were told that at time t_0, a length L had been unwound. $L = R\theta(t_0)$, so:

$$\begin{aligned}\frac{L}{R} =& \frac{FR}{2I_0} t_0^2\\ t_0^2 =& \frac{2LI_0}{FR^2}\\ t_0 =& \sqrt{\frac{2LI_0}{FR^2}}\\ \omega(t_0) = \omega_0 =& \frac{FR}{I_0} t_0\\ =& \frac{FR}{I_0} \frac{1}{R} \sqrt{\frac{2LI_0}{F}}\\ \omega_0^2 =& \frac{F^2}{I_0^2} \frac{2LI_0}{F}\\ I_0 =& \frac{2FL}{\omega_0^2}\\ =& \frac{2 \cdot 5m \cdot 10N}{(0.5s^{-1})^2}\\ =& 400kgm^2\end{aligned}$$

6.12

Since the system is initially at rest, to keep it from rotating once the masses have been released we must force

$$\sum \vec{\tau} = \vec{0}$$

Luckily for us, there are only two torques exerted on the beam - one from the left mass, and one from the Atwood machine. Let's call the tension in the rope to the Atwood machine T. Then:

$$\begin{aligned} 0 =& \sum \tau \\ =& M_1 g \ell_1 - T\ell_2 \end{aligned}$$

Now we need to figure out what T is - we can do this using the Atwood machine and balancing forces. We'll call the tension in the rope going over the pulley T'.

$$\begin{aligned} M_2 a =& \sum F_2 \\ =& M_2 g - T' \\ a =& g - \frac{T'}{M_2} \\ M_3 a =& \sum F_1 \\ =& T' - M_3 g \\ a =& \frac{T'}{M_3} - g \\ g - \frac{T'}{M_2} =& \frac{T'}{M_3} - g \\ 2g =& T' \left(\frac{1}{M_2} + \frac{1}{M_3} \right) \\ T' =& \frac{2gM_2M_3}{M_2 + M_3} \end{aligned}$$

We see by looking at the pulley that $T = 2T'$ because the pulley doesn't accelerate.

$$\begin{aligned} T =& \frac{4gM_2M_3}{M_2 + M_3} \\ 0 =& M_1 g \ell_1 - \frac{4gM_2M_3}{M_2 + M_3} \ell_2 \\ M_1 \ell_1 =& \frac{4M_2M_3\ell_2}{M_2 + M_3} \end{aligned}$$