HW 3 Solutions

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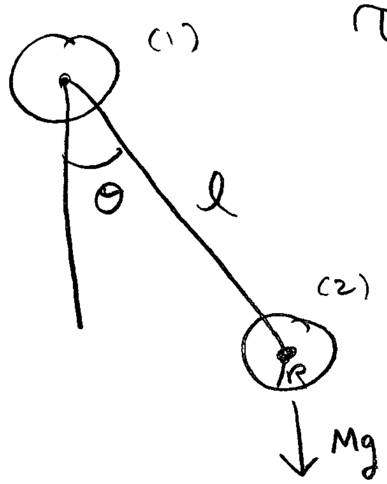
KK 6.15



$$\tau = I\ddot{\theta} = -Mgl\sin\theta$$

for small θ, $\sin\theta \sim \theta$, and

$$\ddot{\theta} + \omega^2\theta = 0 \text{ with } \omega = \sqrt{\frac{Mgl}{I}}.$$

The period is $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgl}}}$.

The moment of inertia is

$$I = I_1 + I_2$$

$$= \left(\tfrac{1}{2}MR^2\right) + \underbrace{\left(\tfrac{1}{2}MR^2 + Ml^2\right)}_{\text{parallel axis theorem}}$$

$$\boxed{I = M(R^2 + l^2)}$$

KK 6.16

This is similiar to the above, but now I is different b/c the first disk has been replaced by a pivot. So,

$\omega = 2\pi\sqrt{\frac{Mgl}{I}}$ as before, but with $I = \frac{1}{2}MR^2 + Ml^2$.

Since $T \propto \frac{1}{\omega}$, minimizing T corresponds to maximizing ω.

$\omega \propto \dfrac{l}{\frac{R^2}{2} + l^2}$. The extrema satisfy $\dfrac{d\omega}{dl} = 0$,

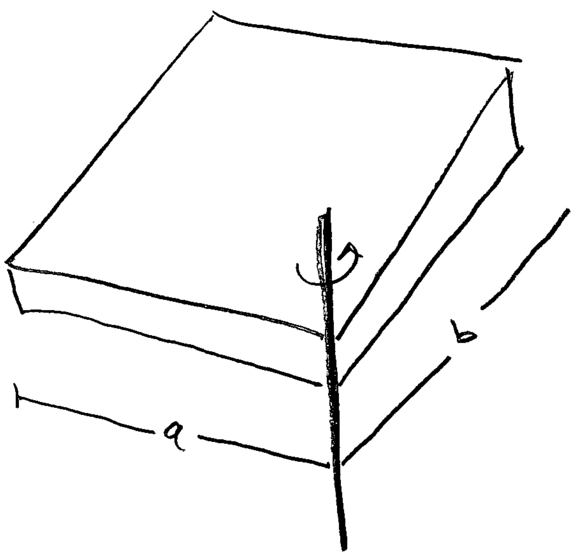
KK 6.16 con't

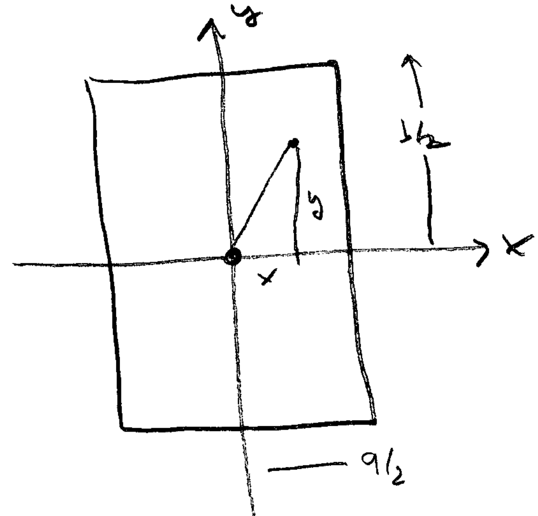
$$\frac{dw}{dl} = \frac{d}{dl}\left(\frac{l}{R^2/2+l^2}\right) = \frac{1}{R^2/2+l^2} - \frac{l}{(R^2/2+l^2)^2}\,2l$$

$$= \frac{R^2/2 - l^2}{(R^2/2+l^2)^2} = 0 \Rightarrow \boxed{l = R/\sqrt{2}}$$

HRK 12.6

First, calculate the moment of inertia about the center:





An arbitrary point on the large face is a distance $\sqrt{x^2+y^2}$ away from origin (axis of rotation). So,

$$I_z = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_0^c dz\,(x^2+y^2)\,dm, \quad dm = \rho\,dV = \frac{M}{abc}\,dx\,dy\,dz$$

$$= \frac{M}{abc}\int_0^c dz\; 4\int_0^{a/2} dx \int_0^{b/2} dy\,(x^2+y^2)$$

$$= \frac{4M}{ab}\int_0^{a/2} dx \int_0^{b/2} dy\,(x^2+y^2)$$

HRK 12.6 con't

$$I_z = \frac{4M}{ab}\left(\int_0^{b/2} dy \int_0^{a/2} dx\, x^2 + \int_0^{a/2} dx \int_0^{b/2} dy\, y^2 \right)$$

$$= \frac{4M}{ab}\left(\frac{b}{2}\frac{1}{3}\left(\frac{a}{2}\right)^3 + \frac{a}{2}\frac{1}{3}\left(\frac{b}{2}\right)^3 \right) = \frac{4M}{ab}\left(\frac{a^3 b}{3\cdot 16} + \frac{a b^3}{3\cdot 16} \right)$$

$$= \frac{M}{12}(a^2+b^2) .$$

Now use the parallel axis Theorem to translate axis to corner,

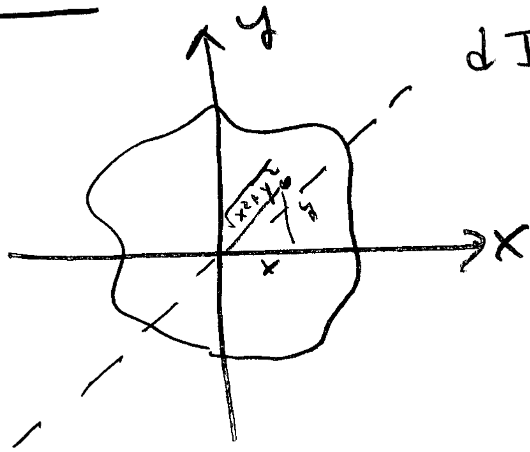
$$I_{corner} = I_z + M\left(\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \right)$$

$$= \frac{M}{12}(a^2+b^2) + \frac{M}{4}(a^2+b^2)$$

$$\boxed{I = \frac{M}{3}(a^2+b^2)}$$

HRK 12.9

(a)



$dI_x = dm\, y^2$,

$dI_y = dm\, x^2$, and so

$$dI_x + dI_y = dm(x^2+y^2)$$

$$= dm\, r^2$$

distance to z-axis

$$= dI_z \ \checkmark$$

HRK 12.9 con't

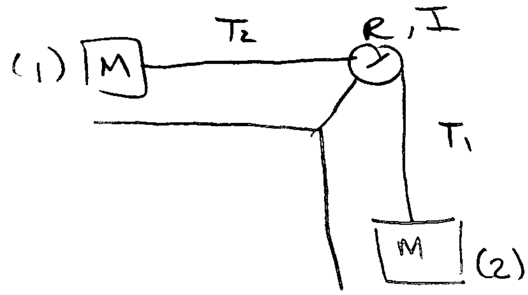
(b) for a circle, $dI_z = dm\,r^2 = dI_x + dI_y$, but $dI_x = dI_y$ by symmetry, so $dI_{\substack{any\\diameter}} = \frac{1}{2} dI_z$.

$$I_{\substack{any\\diameter}} = \frac{1}{2} I_z = \frac{1}{2}\left(\frac{M}{\pi R^2}\right)\int (dr\,d\phi\,r) r^2 = \frac{1}{2}\left(\frac{M}{\pi R^2}\right)(2\pi)\frac{R^4}{4}$$

$$= \frac{MR^2}{4}$$

$$\boxed{I_{\substack{any\\diameter}} = \frac{MR^2}{4} = \frac{1}{2} I_z}$$

HRK 12.29



a = const., we are told.

no slip means $R\alpha = a$, so $\alpha = \frac{a}{R}$.

So $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

$= \frac{1}{2}\frac{a}{R}t^2$,

So, $\boxed{a = \frac{2R\theta}{t^2}, \quad \alpha = \frac{2\theta}{t^2}}$

summing up the forces on block (2),

$$\sum_{(2)} F_y = ma = mg - T_1 .$$

Also, the angular acceleration of the pulley is caused by a torque,

$\tau = R(T_1 - T_2) = I\alpha$, so

$$\boxed{\begin{aligned} T_1 &= m(g-a) \\ &= m\left(g - \frac{2R\theta}{t^2}\right)\end{aligned}}$$

and $-\frac{I\alpha}{R} + T_1 = T_2$, so

$$T_2 = m\left(g - \frac{2R\theta}{t^2}\right) - \frac{2I\theta}{t^2 R}$$

$$\boxed{T_2 = mg - \frac{2Rm\theta}{t^2} - \frac{2I\theta}{t^2 R} = mg - \frac{2\theta}{t^2}\left[mR + \frac{I}{R}\right]}$$

HRK 12.34

energy is conserved, so $E_i = E_f$.

E_i is purely potential energy, $E_i = mgh$

E_f is purely kinetic (choosing to measure our height from this point),

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2} I_{\substack{\text{sphere}\\ \text{shell}}} \omega^2_{\substack{\text{sphere}\\ \text{shell}}} + \frac{1}{2} I_{\text{pulley}} \omega^2_{\text{pulley}}$$

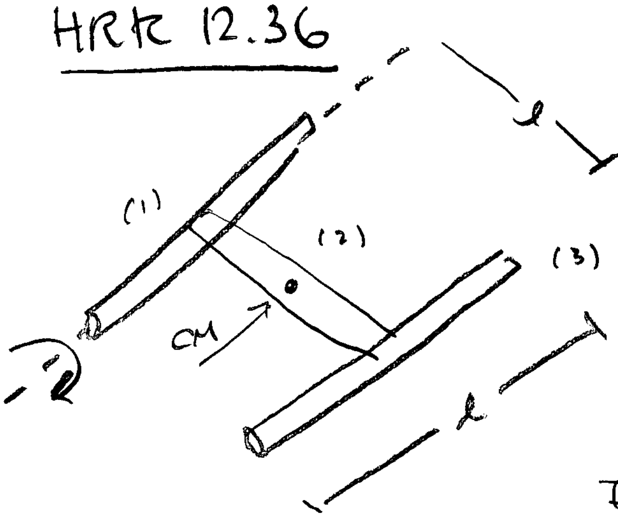
using $\omega_{\substack{\text{sphere}\\ \text{shell}}} = \frac{v}{R}$, $\omega_{\text{pulley}} = \frac{v}{r}$, and

$$I_{\substack{\text{sphere}\\ \text{shell}}} = \frac{2MR^2}{3}, \quad I_{\text{pulley}} = ?,$$

we have $E_i - E_f = 0 \Rightarrow$

$$\boxed{v = \sqrt{\frac{2mgh}{m + \frac{2}{3}M + I_P/r^2}}}$$

HRK 12.36



$I = I_1 + I_2 + I_3$

$I_1 = 0$ b/c the rod is aligned w/ the axis. Every point on the rod is rotating about the axis at a very small distance, which is zero in the limit where the rod is infinitely thin.

$I_2 = \frac{1}{3}ML^2$

$I_3 = I_{CM} + ML^2 = \frac{4}{3}ML^2$

HRK 12.36 con't

$E_i = E_f \Rightarrow$

$\frac{1}{2} I \omega^2 = M_{tot}\, g\, \Delta h_{cm}$, $M_{tot} = 3M$, $\Delta h_{cm} = \frac{L}{2}$, and

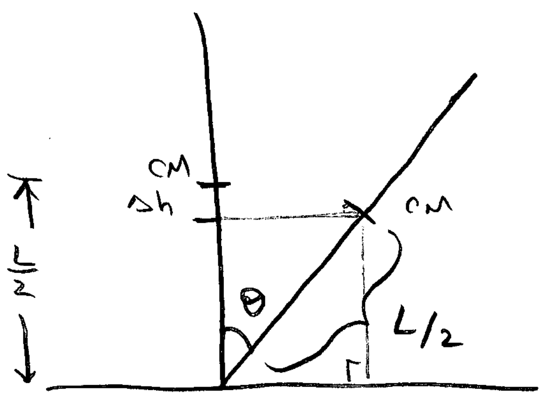
$$\omega^2 = \frac{2 \cdot 3 \cdot \cancel{M} \cdot g \cdot \frac{\cancel{L}}{2}}{\frac{4}{3} \cancel{M} L^{\cancel{2}}} = \frac{9}{4}\frac{g}{L}, \text{ and}$$

$$\boxed{\omega = \frac{3}{2}\sqrt{\frac{g}{L}}}$$

HRK 12.38

Energy conservation: $E_i = 0 = E_f = \frac{1}{2} I \omega^2 - mg\,\Delta h_{cm}$

want Δh_{cm} in terms of θ:



$\frac{L}{2} - \Delta h = \frac{L}{2}\cos\theta$, so

$\Delta h = \frac{L}{2}(1 - \cos\theta)$.

So we can use this to find ω now:

$$\omega^2 = \frac{2mg\,\Delta h_{cm}}{I}$$

$$= \frac{mgL(1-\cos\theta)}{I}$$

here $I = \frac{1}{3} M L^2$, so

$$\omega^2 = \frac{3g}{L}(1 - \cos\theta)$$

HRK 12.38 con't

$$\alpha = \frac{d}{dt}\dot{\theta} = \frac{d}{dt}\omega = \frac{1}{2\omega}\frac{d}{dt}\omega^2 = \frac{1}{2\omega}\frac{3g}{L}\frac{d}{dt}(1-\cos\theta)$$

$$\alpha = \frac{3g}{2L}\sin\theta.$$

We want to know

(a) radial acceleration

$$\boxed{a_r = L\omega^2 = 3g(1-\cos\theta)}$$

(b) tangential acceleration,

$$\boxed{a_t = L\alpha = \frac{3g}{2}\sin\theta}$$

(c) if $a_t = g$, Then $\sin\theta = \frac{2}{3} \Rightarrow \boxed{\theta \sim 0.73 \text{ or } 42^\circ}$

(d) The top end is falling faster Than a freely falling body, which can only happen as a result of internal stresses which can break The chimney.

HRR 12.39 Let earth be a uniform sphere,

$$I = \frac{2}{5} M R^2$$

$$KE = \frac{1}{2} I \omega^2 = \cancel{\frac{1}{2}} \frac{\cancel{2} M R^2}{5} \left(\frac{2\pi}{T}\right)^2$$

$$\frac{d(KE)}{dt} = \frac{4\pi^2}{5} M R^2 \frac{d}{dt} T^{-2}$$

$$= \frac{4\pi^2}{5} M R^2 \left(\frac{-2}{T^3}\right) \frac{dT}{dt} \quad . \quad \text{plugging in the \#'s,}$$

(a) $$\boxed{\frac{dK}{dt} = -1.88 \cdot 10^{12} \ \frac{J}{s}}$$

(b) average acceleration is

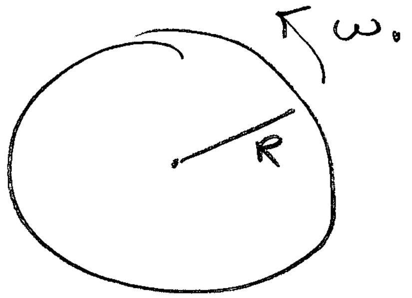
$$\alpha = \frac{d\omega}{dt} = 2\pi \frac{d}{dt} T^{-1} = -\frac{2\pi}{T^2} \frac{dT}{dt}$$

$$\boxed{\alpha = 2.67 \cdot 10^{-22} \ \frac{rad}{s^2}}$$

(c) $$\tau = FR = I\alpha \ , \text{ so}$$

$$\boxed{F = \frac{I\alpha}{R} = 4.06 \cdot 10^{9} N}$$

HRK 12.40



(a)

$$d\tau = -r\, dF$$

$dF = f\,dA$, f is the force density,

$$f = \frac{F}{A} = \frac{\mu N}{A} = \frac{\mu M g}{\pi R^2}.$$

So,

$$\tau = \int d\tau = -\int_0^{2\pi} d\phi \int_0^R dr\, r^2 f$$

$$= -2\pi f \frac{R^3}{3} = -\frac{2}{3}\mu M g R$$

(b)

$\tau = I\alpha$, so $\alpha = \dfrac{\tau}{I}$ (const), so

$\omega = \omega_0 + \alpha t$. at $t = t_f$, let $\omega = 0$. Then

$0 = \omega_0 + \alpha t_f$, and

$$t_f = \frac{-\omega_0}{\alpha} = \frac{-\omega_0 I}{\tau} = \frac{-\omega_0\left(\frac{1}{2}\cancel{M}R^{\cancel{2}}\right)}{-\frac{2}{3}\mu \cancel{M} g \cancel{R}}$$

$$\boxed{t_f = \frac{3\omega_0 R}{4\mu g}}$$