Phys 21 HW 5 Solutions

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I. HRK 15.17

$$F(r) = -\frac{a}{r^2} + \frac{b}{r^3}.$$

a) The separation at equlibrium is the separation for which $F(r) = 0$. This is given by $r = \frac{b}{a}$. At

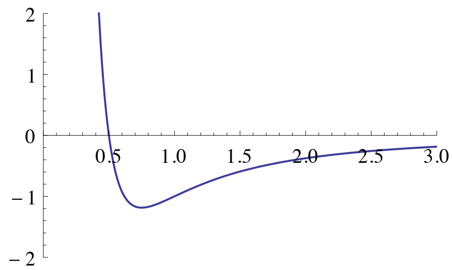

FIG. 1. F(r) for a=2, b=1

this radius the attractive and repulsive forces are balanced and the net force vanishes. This is the equlibrium position.

b) Expanding the force about the equilibrium position yields

$$F(r) \approx -\frac{a^4\left(r - \frac{b}{a}\right)}{b^3} + \frac{3a^5\left(r - \frac{b}{a}\right)^2}{b^4} + O\left(\left(r - \frac{b}{a}\right)^3\right)$$

Neglecting all but the leading term, this is just Hook's law, $m\ddot{x} = -kx$, with $x = r - b/a$, and $k = a^4/b^3$

c) The solution to Hook's equation is of the form $x(t) = \cos(\sqrt{k/m}t)$, and the period is $T = 2\pi/\sqrt{k/m}$. For the problem at hand, this is

$$T = \frac{2\pi b^{3/2} m^{1/2}}{a^2},$$

where m is the mass of the atoms in question.

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II. HRK 15.20

For two particles undergoing simple harmonic motion about the same origin, and with the same frequency and amplitude, the positions are given by

$$x_1(t) = A\cos(\omega t + \phi_1), \qquad x_2(t) = A\cos(\omega t + \phi_2).$$

At $t = t_*$, the two positions are the same and equal to $A/2$, so we have

$$x_1(t_*) = x_2(t_*) = A/2$$
$$\cos(\omega t_* + \phi_1) = \cos(\omega t_* + \phi_2) = \frac{1}{2}$$
$$\omega t_* + \phi_1 = \pm\frac{\pi}{3}, \qquad \omega t_* + \phi_2 = \pm\frac{\pi}{3}$$

The $\pm$ signs are important, as we'll see shortly. I'll fix the first one to be positive and the second one I'll keep arbitrary. So we have

$$\omega t_* + \phi_1 = \frac{\pi}{3}, \qquad \omega t_* + \phi_2 = \pm\frac{\pi}{3}$$

The velocities are given by

$$v_1(t_*) = -A\omega\sin(\frac{\pi}{3}) = -A\omega\frac{\sqrt{3}}{2}, \qquad v_2(t_*) = -A\omega\sin(\pm\frac{\pi}{3}) = \mp A\omega\frac{\sqrt{3}}{2}$$

So when we choose the opposite signs, the velocities at the time of crossing are opposite signs. So,

$$(\omega t_* + \phi_1) - (\omega t_* + \phi_2) \equiv \Delta\phi = \frac{2\pi}{3}.$$

III. HRK 15.21

The block's equation of motion is

$$m\ddot{x} = \sum F = F_1 + F_2 = -k_1 x - k_2 x = -(k_1 + k_2)x$$

So this is just the equation of motion for a single spring system with spring constant $k = k_1 + k_2$. So

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}$$

Since if only spring 1 or 2 were attached to the block the frequency would be $\nu_{1,2} = (2\pi)^{-1}\sqrt{k_{1,2}/m}$, we can also write

$$\nu = \sqrt{\nu_1^2 + \nu_2^2}.$$

IV. HRK 15.22

For this problem, let the block be at position x and $x_{1,2}$ be the displacement distance of the springs from their equilibrium value. The block will feel an effective force

$$F = -k_{\text{eff}}(x_1 + x_2).$$

The second sping exerts a force $F = -kx_2$ on the block and on the first spring, and the first spring exerts a force $F = -kx_1$ on the second spring. All these forces are equal since springs' centers of mass are not accelerating. So we have

$$-k_1 x_1 = -k_2 x_2 = -k_{\text{eff}}(x_1 + x_2).$$

Using $x_1 = k_2 x_2 / k_1$, we can see that

$$k_2 x_2 = -k_{\text{eff}}\Big(\frac{k_2}{k_1} + 1\Big)x_2 \Rightarrow k_{\text{eff}} = \Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)^{-1} = \frac{k_1 k_2}{k_1 + k_2}$$

So that

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}} = \frac{\nu_1 \nu_2}{\sqrt{\nu_1^2 + \nu_2^2}}.$$

V. HRK 15.34

When the bullet hits the block, some energy is lost to heat, but linear momentum is conserved. Then the block begins to execute simple harmonic motion, and energy is converved. Assuming the spring is massless, the initial velocity of the block is found by

$$mv = (M + m)v_0 \qquad \Rightarrow \qquad v_0 = \frac{m}{M + m}v.$$

Since the spring is initially at its equilibrium position, the energy of the system (initially, and for all time) is

$$E = \frac{1}{2}(M + m)v_0^2.$$

Basically the bullet is supplying us with initial conditions for the simple harmonic oscillator equation, which now reads

$$x(t) = A\sin(\omega t),$$

with $\omega = \sqrt{k/(M + m)}$. The initial velocity is $v_0 = A\omega$, so we have that

$$A = \frac{v_0}{\omega} = v_0\sqrt{\frac{M + m}{k}} = \frac{m}{\sqrt{k(M + m)}}v.$$

We could have also said that the energy at the maximum displacement is

$$E = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}(M+m)v_0^2,$$

and that $(\Delta x) = A$. So

$$A = \sqrt{\frac{(M+m)}{k}}\frac{m}{(M+m)}v = \frac{m}{\sqrt{k(M+m)}}v$$

VI. HRK 15.37

Once the system is released from rest, energy is conserved. At the starting point, all the energy is tied into potential energy, and as the spring returns to its equilibrium position, all that energy is converted into kinetic. We have

$$E = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}Mv_{\rm cm}^2 + \frac{1}{2}I\omega^2.$$

The cylinder has $I = MR^2/2$, and the no slipping condition is $v = R\omega$. So then,

$$E = \frac{1}{2}k(\Delta x)^2 = \frac{3}{4}Mv_{\rm cm}^2,$$

so that

$$v_{\rm cm} = \sqrt{\frac{2}{3}\frac{k}{M}}(\Delta x).$$

a),b) The translational and rotational energies are then

$$K_{\rm trans} = \frac{1}{2}Mv_{\rm cm}^2 = \frac{1}{3}k(\Delta x)^2 = 5.6 \text{ Joules}$$

$$K_{\rm rot} = \frac{1}{2}I\omega^2 = \frac{1}{6}k(\Delta x)^2 = 2.8 \text{ Joules}$$

c) That the center of mass of the cylinder undergoes simple harmonic motion can be seen as follows: $(\Delta x) = R\theta$, and $v_{\rm cm} = R\omega$, so that the energy at a generic moment in time is

$$E = \frac{1}{2}kR^2\theta^2 + \frac{3}{4}MR^2\dot{\theta}^2.$$

Since this is conserved, $\partial_t E = 0$, and

$$k\theta + \frac{3}{2}M\ddot{\theta} = 0 \qquad \Rightarrow \qquad \ddot{\theta} + \tilde{\omega}^2\theta = 0.$$

This is the equation for a SHO with

$$T = \frac{2\pi}{\tilde{\omega}} = 2\pi\sqrt{\frac{3M}{2k}}.$$

VII. HRK 15.52

a), b) Start the particle from rest anywhere inside the bowl, but orient the bowl so that it's only moving in the x-direction. Let θ be the angle between the line normal to the bowl at the bowl's bottom, and the radial vector pointing towards the particle. The max height is found as

$$\cos\theta = 1 - \frac{h}{R}, \qquad h = R(1-\cos\theta).$$

For small θ, the heigh is $h \approx R\theta^2/2$. The energy is

$$E = mgh + \frac{1}{2}mv^2 = mgR(1-\cos\theta) + \frac{1}{2}mR^2\dot{\theta}^2.$$

For small θ, i.e. near the bottom of the bowl, this is

$$E \approx \frac{mgR}{2}\theta^2 + \frac{1}{2}mR^2\dot{\theta}^2.$$

Since this is conserved, $\partial_t E = 0$, and we have

$$\ddot{\theta} + \frac{g}{R}\theta = 0.$$

This is just a SHO with

$$\frac{g}{R} = \frac{g}{l},$$

so $l_{\text{eff}} = R$, this *is* just a simple pendulum.

VIII. HRK 15.56

This problem asks us to find the period of oscillation for two different physical pendulua. The general formula is

$$T = 2\pi\sqrt{\frac{I}{MgL}},$$

where L is the distance from the center of mass to the pivot, which in this case is $2r_1 + \Delta$, where r_1 is the inner radius and Δ is the thickness (the Δ comes about because the pivot lies on the inside of one of the pipes). So our job is to calculate the moment of inertia for these two configurations.

The moment of inertia about the center of mass for a constant density annulus (a ring with non-zero thickness) is

$$dI = r^2 dm = r^2\rho dA = r^2\rho r dr d\phi \qquad \Rightarrow \qquad I = (2\pi)\rho\frac{r^4}{4}\Bigg|_{r_1}^{r_1+\Delta} = \frac{\pi\rho}{2}((r_1+\Delta)^4 - r_1^4).$$

The density is $\rho = M/A$, and

$$A = \pi((r_1 + \Delta)^2 - r_1^2).$$

Since this expression is rather complicated, I'll just plug in the numbers now, to find that

$$I_{\text{cm}} = 110.8M\text{cm}^2$$

a) The moment of inertia about the pivot is

$$I_a = I_1 + I_2 = \left(I_{\text{cm}} + Mr_1^2\right) + \left(I_{\text{cm}} + M(3r_1 + 2\Delta)^2\right) = 2I_{\text{cm}} + M(r_1^2 + (3r_1 + 2\Delta)^2) = 1341.9M\text{cm}^2.$$

So then the period is

$$T_a = (2\pi)\sqrt{\frac{1341.9\text{cm}^2}{gL}} = 1.602\text{s}.$$

b) For part b), the only difference is in the second I_{cm}. Because of the perpendicular axis theorem, and the symmetry of the problem, the moment of inertia about an axis in the plane of the cylinder should be 1/2 as much as through the z-axis. So the formula for the total I changes to

$$I_b = I_1 + I_2 = \left(I_{\text{cm}} + Mr_1^2\right) + \left(\frac{I_{\text{cm}}}{2} + M(3r_1 + 2\Delta)^2\right) = \frac{3}{2}I_{\text{cm}} + M(r_1^2 + (3r_1 + 2\Delta)^2) = 1286.5M\text{cm}^2.$$

The new period is

$$T_b = (2\pi)\sqrt{\frac{1286.5\text{cm}^2}{gL}} = 1.569\text{s}.$$

And lastly,

$$1 - \frac{T_b}{T_a} = 0.021,$$

So that the new period is 2% less than the old.

IX. HRK 15.61

See the mathematica notebook for this one.

X. HRK 15.64

The solution to a damped harmonic oscillator equation,

$$m\ddot{x} + b\dot{x} + kx = 0,$$

for the underdamped case ($\omega_0^2 > \gamma^2$) is

$$x(t) = Ae^{-\gamma t}\cos(\omega_d t + \phi),$$

where

$$\gamma = \frac{b}{2m}, \qquad \omega_0^2 = \frac{k}{m}, \qquad \omega_d^2 = \omega_0^2 - \gamma^2.$$

So the underdamped case corresponds to $\omega_d \in \mathbb{R}$ (it's a real number). The period is given by $T = 2\pi/\omega_d$. We are told that after 4 cycles the amplitude falls to 3/4 it's original value, which in equations translates to

$$\frac{x(4T)}{x(0)} = e^{-4T\gamma}\frac{\cos(\omega_d(4T)+\phi)}{\cos\phi} = e^{-4T\gamma} = \frac{3}{4}.$$

In the above I used the fact that $\cos(4T\omega_d + \phi) = \cos(4T\omega_d)\cos\phi - \sin(4T\omega_d)\sin\phi = \cos\phi$.

a) I can now solve for b. Taking the logarithm of the above equation, I have

$$-4T\gamma = -4\frac{2\pi}{\sqrt{\omega_0^2-\gamma^2}}\gamma = \ln(3/4) = -\ln(4/3).$$

Squaring, and substituting in the values for ω_0, γ, this becomes

$$\frac{8\pi}{\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}}\frac{b}{2m} = \ln(4/3).$$

This is perverse, so I'm just going to plug it into my handy calculator. Throwing in m, k, we can solve for b:

$$b \to 0.112292\text{kg/s}$$

b) How much energy has been lost? Well the amplitude has decreased to 3/4 it's original value, so it's amplitude has decreased to $(3/4)^2 = 9/16$ it's original value. They tell us it's original value, so we can find the energy lost as

$$\left(1 - \frac{9}{16} = \frac{7}{16}\right)E_i = \frac{7}{16}\times\frac{1}{2}k(\Delta x)^2 = \frac{7}{32}(12.6N/m)(26.2\text{cm})^2 = 0.189 \text{ Joules}.$$

Phys 21 HW 5

by Gavin Hartnett

HRK 15.17

HRK 15.56

HRK 15.61

Part a)

Here are some Lissajous figures for xm=ym and ϕx=ϕy: The figures depend on ϕ, so I've done a few different ones.

In[1]:=
```
Clear["Global`*"];
```

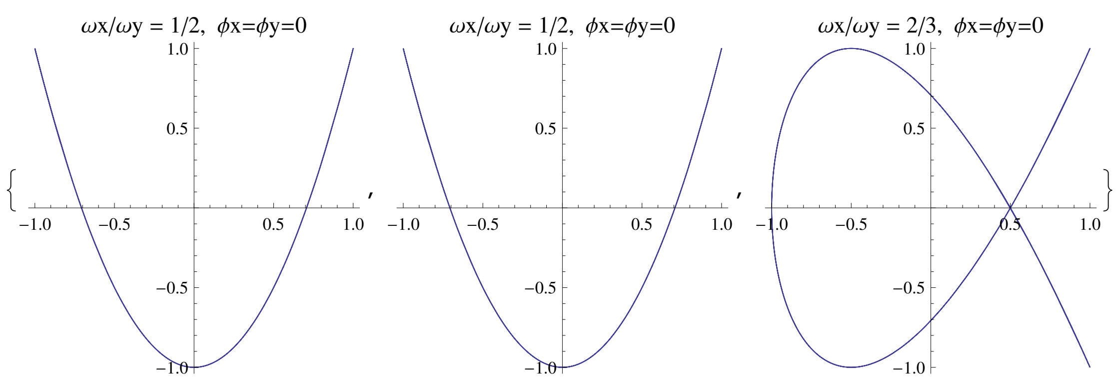
In[2]:=
```
x[t_] = Cos[ωx t + ϕx];
y[t_] = Cos[ωy t + ϕy];
```

In[4]:=
```
ϕx = ϕ;
ϕy = ϕ;
```

In[6]:=
```
ωxlist = {1, 1, 2};
ωylist = {2, 2, 3};
```

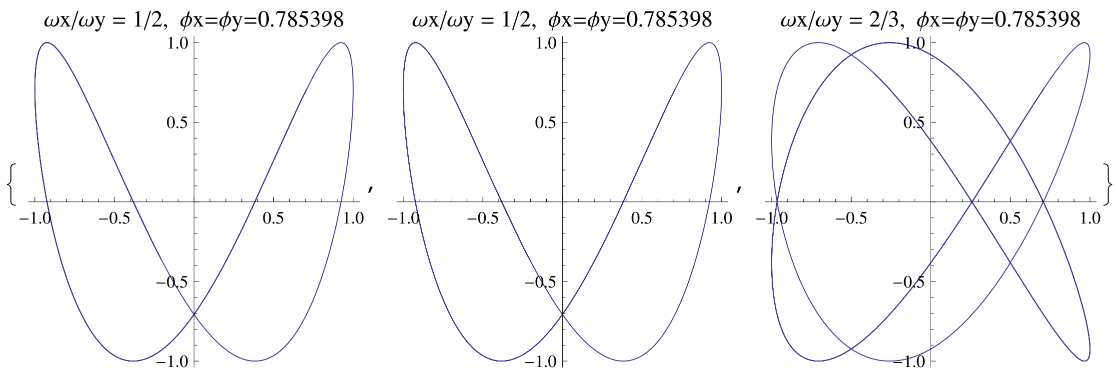
In[8]:=
```
ϕ = 0;
Table[ParametricPlot[{x[t] /. ωx → ωxlist[[ii]], y[t] /. ωy → ωylist[[ii]]}, {t, 0, 10},
  PlotRange → All, PlotLabel → StringJoin["ωx/ωy = ", ToString[ωxlist[[ii]]],
   "/", ToString[ωylist[[ii]]], ",  ϕx=ϕy=", ToString[ϕ]]], {ii, 1, 3}]
```

Out[9]=



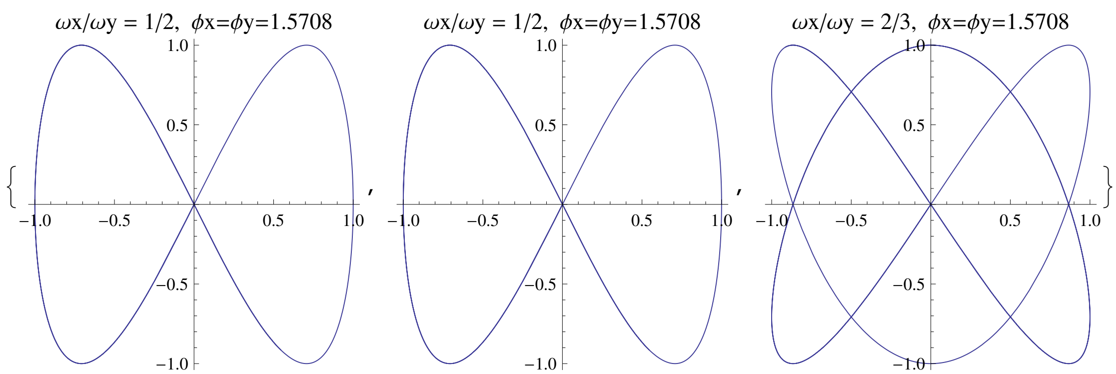
In[10]:=
```
ϕ = N[π / 4];
Table[ParametricPlot[{x[t] /. ωx → ωxlist⟦ii⟧, y[t] /. ωy → ωylist⟦ii⟧}, {t, 0, 10},
  PlotRange → All, PlotLabel → StringJoin["ωx/ωy = ", ToString[ωxlist⟦ii⟧],
    "/", ToString[ωylist⟦ii⟧], ",   ϕx=ϕy=", ToString[ϕ]]], {ii, 1, 3}]
```

Out[11]=



In[12]:=
```
ϕ = N[π / 2];
Table[ParametricPlot[{x[t] /. ωx → ωxlist⟦ii⟧, y[t] /. ωy → ωylist⟦ii⟧}, {t, 0, 10},
  PlotRange → All, PlotLabel → StringJoin["ωx/ωy = ", ToString[ωxlist⟦ii⟧],
    "/", ToString[ωylist⟦ii⟧], ",   ϕx=ϕy=", ToString[ϕ]]], {ii, 1, 3}]
```

Out[13]=



Part b)

Here are some Lissajous figures for xm=ym.

In[14]:=
```
Clear["Global`*"];
```

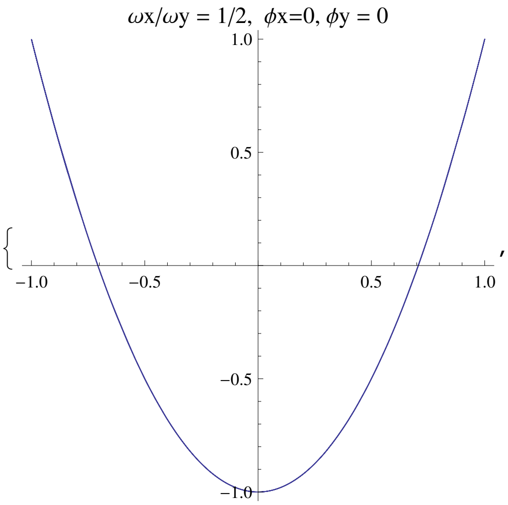
In[15]:=
```
x[t_] = Cos[ωx t + ϕx];
y[t_] = Cos[ωy t + ϕy];
```

In[17]:=
```
ϕx = ϕ;
ϕy = 0;
```

In[19]:=
```
ωxlist = {1, 1, 2};
ωylist = {2, 2, 3};
```

In[21]:=
```
ϕ = 0;
Table[ParametricPlot[{x[t] /. ωx → ωxlist⟦ii⟧, y[t] /. ωy → ωylist⟦ii⟧},
  {t, 0, 10}, PlotRange → All, PlotLabel →
   StringJoin["ωx/ωy = ", ToString[ωxlist⟦ii⟧], "/", ToString[ωylist⟦ii⟧],
    ",  ϕx=", ToString[ϕ], ", ϕy = 0"], ImageSize → 250], {ii, 1, 3}]
```

Out[22]= {



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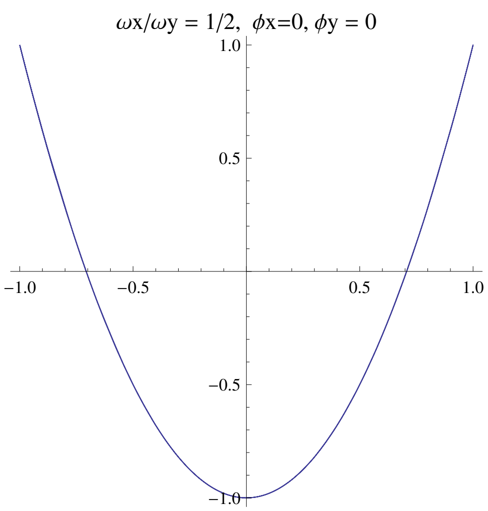

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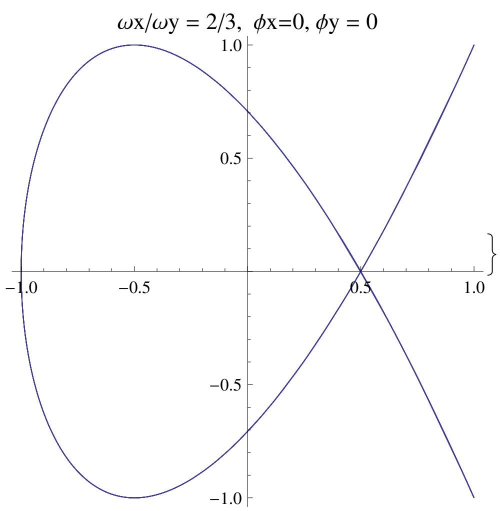
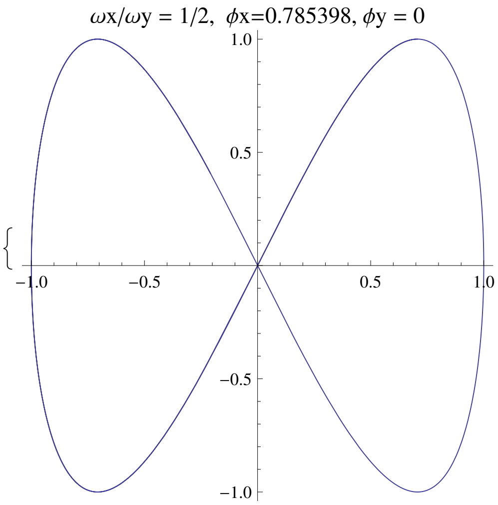


}

In[23]:=
```
ϕ = N[π / 4];
Table[ParametricPlot[{x[t] /. ωx → ωxlist⟦ii⟧, y[t] /. ωy → ωylist⟦ii⟧},
  {t, 0, 10}, PlotRange → All, PlotLabel →
   StringJoin["ωx/ωy = ", ToString[ωxlist⟦ii⟧], "/", ToString[ωylist⟦ii⟧],
    ",  ϕx=", ToString[ϕ], ", ϕy = 0"], ImageSize → 250], {ii, 1, 3}]
```

Out[24]= {



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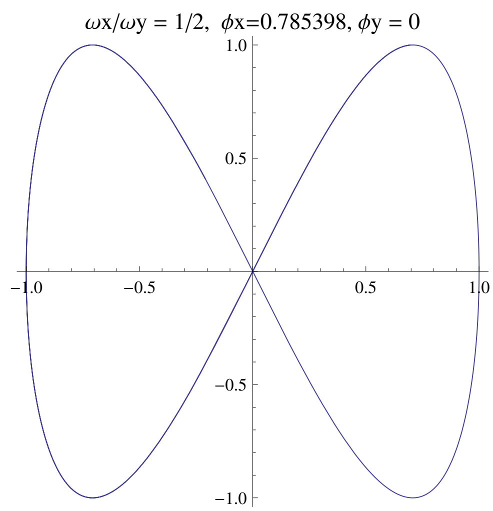

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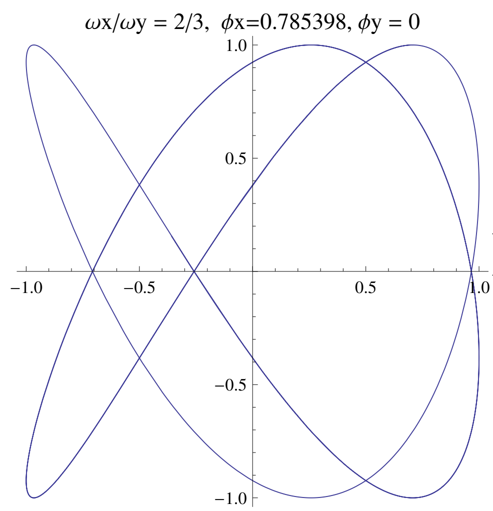
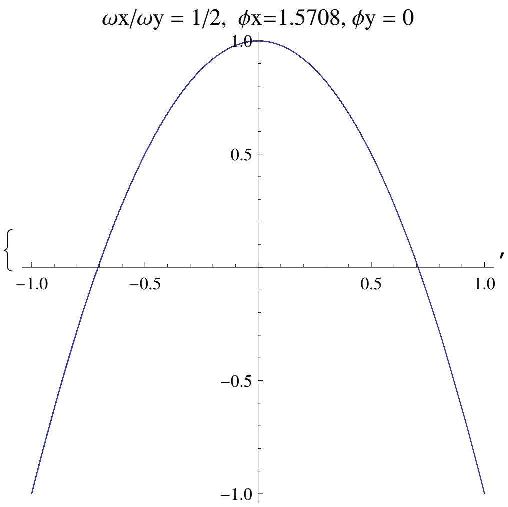


}

In[25]:=
```
ϕ = N[π / 2];
Table[ParametricPlot[{x[t] /. ωx → ωxlist⟦ii⟧, y[t] /. ωy → ωylist⟦ii⟧},
  {t, 0, 10}, PlotRange → All, PlotLabel →
   StringJoin["ωx/ωy = ", ToString[ωxlist⟦ii⟧], "/", ToString[ωylist⟦ii⟧],
    ",  ϕx=", ToString[ϕ], ", ϕy = 0"], ImageSize → 250], {ii, 1, 3}]
```

Out[26]= {



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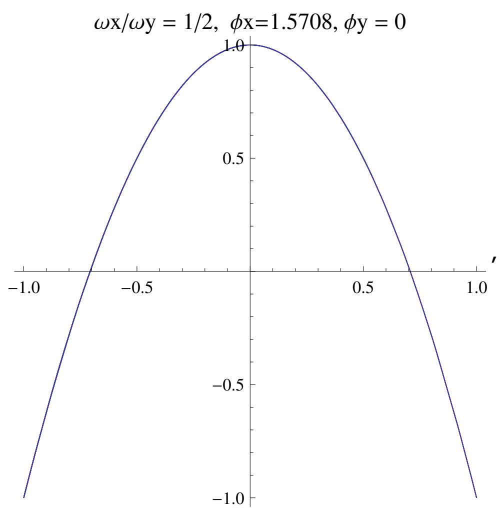

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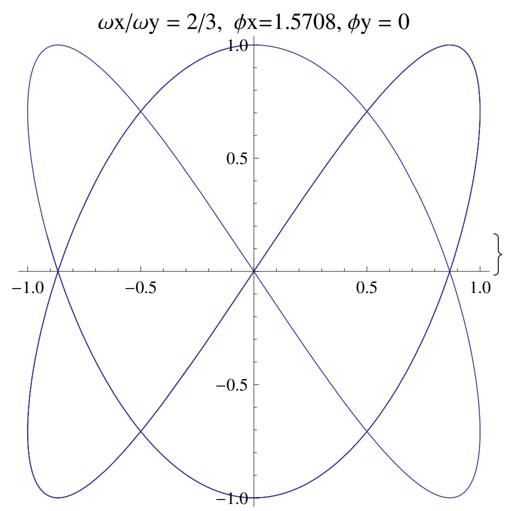


}

Part c)

Now I want to show that when ωx/ωy is not rational, the curves fill the whole unit square.

In[27]:=
```
Clear["Global`*"];
```

In[28]:=
```
x[t_] = Cos[ωx t + ϕx];
y[t_] = Cos[ωy t + ϕy];
```

In[30]:=
```
ϕx = ϕ;
ϕy = ϕ;
```

In[32]:=
```
ωxlist = {1, 1, 2};
ωylist = {2, 2, 3};
```

In[34]:=
```
ωx = N[GoldenRatio]
ωy = 1
```

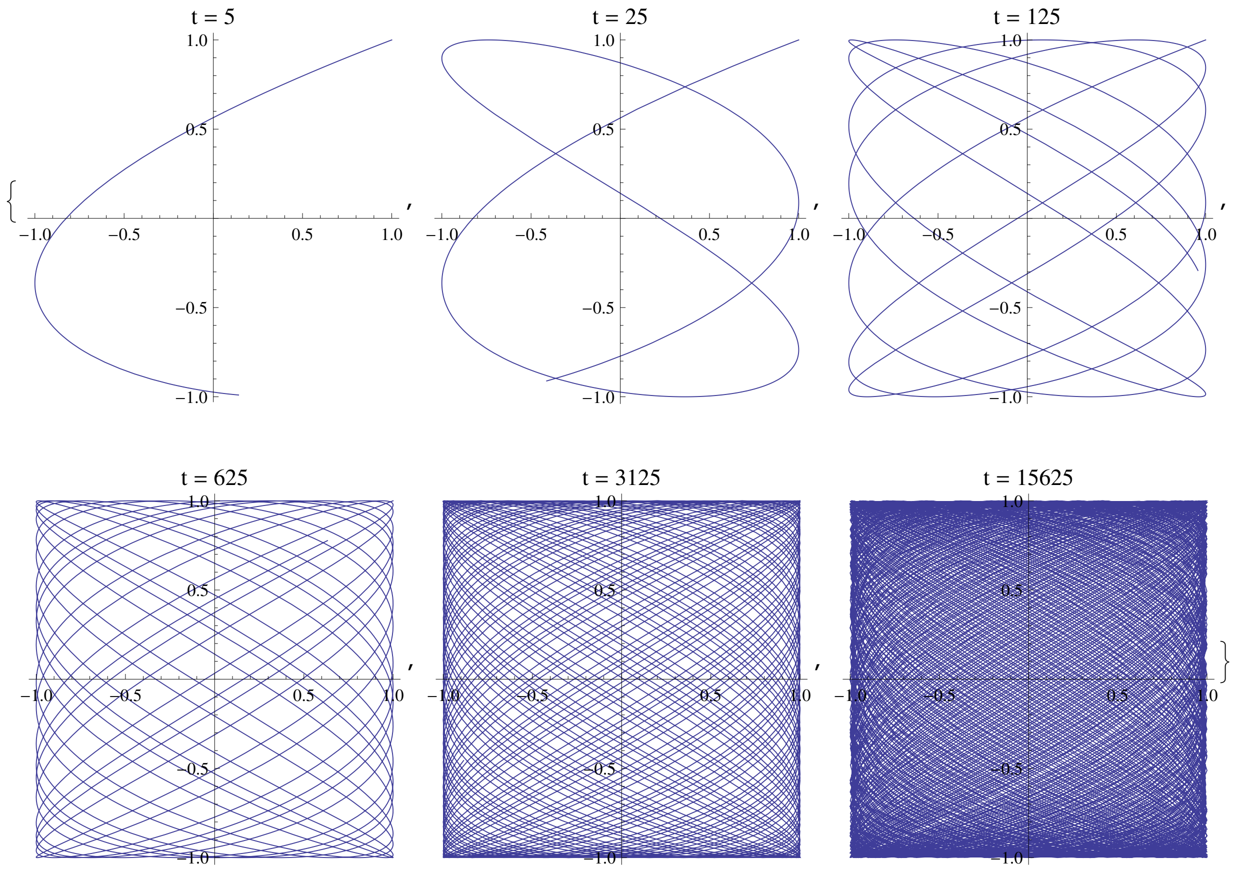
Out[34]= 1.61803

Out[35]= 1

Here's the case of ϕx=0, ϕy=0, and ωx/ωy=Φ (the golden ratio)

In[36]:=
```
ϕ = 0;
Table[ParametricPlot[{x[t], y[t]}, {t, 0, 3^ii}, PlotRange → All,
  PlotLabel → StringJoin["t = ", ToString[5^ii]], ImageSize → 200], {ii, 1, 6}]
```

Out[37]=



Isn't this cool!!!!

HRK 15.64