PHYS 21: Assignment 6 Solutions

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K + K 10.2

$$Q = \frac{\omega}{\gamma}$$

We know that:

$$\omega = \frac{2\pi\text{rad}}{\text{cycle}} \cdot \frac{2\text{cycles}}{\text{second}} = \frac{4\pi\text{rad}}{\text{s}}$$

So:

$$\begin{aligned}
\gamma &= \frac{4\pi\text{rad/s}}{60\text{rad}} = 0.21\frac{1}{\text{s}} \\
b &= \gamma m = 0.063\frac{\text{kg}}{\text{s}} \\
\omega &= \sqrt{\frac{k}{m}} \\
k &= m\omega^2 \\
&= 0.3\text{kg} \cdot (4\pi\text{rad/s})^2 = 47.5\text{N/m}
\end{aligned}$$

K + K 10.5

The equation of motion for the damped oscillator is:

$$0 = \ddot{x} + \gamma\dot{x} + \omega^2 x$$

For $\gamma = 2\omega_0$, we have:

$$0 = \ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x$$

Let's use a trial solution of

$$\begin{aligned}
x(t) &= (A + Bt)e^{-\gamma t/2} \\
&= (A + Bt)e^{-\omega_0 t} \\
\Rightarrow \dot{x} &= -\omega_0(A + Bt)e^{-\omega_0 t} + Be^{-\omega_0 t} \\
\ddot{x} &= \omega_0^2(A + Bt)e^{-\omega_0 t} - 2B\omega_0 e^{-\omega_0 t} \\
0 &= \omega_0^2(A + Bt)e^{-\omega_0 t} - 2B\omega_0 e^{-\omega_0 t} - 2\omega_0^2(A + Bt)e^{-\omega_0 t} + 2\omega_0 Be^{-\omega_0 t} + \omega_0^2(A + Bt)e^{-\omega_0 t} \\
0 &= 2\omega_0^2(A + Bt)e^{-\omega_0 t} - 2\omega_0^2(A + Bt)e^{-\omega_0 t} - 2B\omega_0 e^{-\omega_0 t} + 2\omega_0 Be^{-\omega_0 t} \\
&= 0
\end{aligned}$$

Now we apply the initial conditions:

$$
\begin{aligned}
x(0) =& 0 \\
\Rightarrow 0 =& A \\
\dot{x}(0) =& \frac{I}{m} \\
\Rightarrow \frac{I}{m} =& B \\
\Rightarrow x(t) =& \frac{It}{m} e^{-\omega_0 t}
\end{aligned}
$$

We see that $\dot{x} = 0$ when $t = \frac{2}{\gamma}$, and since the second derivative is always negative this is the maximum.

K + K 10.7

We can see from Equation 10.25 of the textbook that $x(t)$ has the form:

$$
\begin{aligned}
x(t) =& A\cos(\omega t + \varphi) \\
\varphi =& \arctan \frac{\gamma\omega}{\omega_0^2 - \omega^2}
\end{aligned}
$$

So we see that:

$$
\dot{x}(t) = - A\omega \sin(\omega t + \varphi)
$$

To be "in phase" with the driving force, we need

$$
\begin{aligned}
-\sin(\omega t + \varphi) =& \cos(\omega t) \\
-\sin(\omega t)\cos(\varphi) - \sin(\varphi)\cos(\omega t) =& \cos(\omega t)
\end{aligned}
$$

This means that $\sin\varphi = -1$ and $\cos\varphi = 0$ so $\varphi = \frac{3\pi}{2}$. But this forces $\tan\phi = 0$, which occurs when $\omega = \omega_0$, or when resonance occurs.

K + K 10.10

However much power is generated by dropping the weight is dissipated by friction. If the weight has descended a distance L in a time T, then:

$$\begin{aligned}\bar{p} =& \frac{\Delta E}{\Delta t} \\ =& \frac{MgL}{T}\end{aligned}$$

And we'd like to find the energy lost per radian swung:

$$\bar{\Delta E} = \frac{\bar{p}}{\omega}$$

We can also write down the average stored energy

$$\bar{E} = \frac{m}{2}\bar{v^2} + \frac{mg\ell}{2}\bar{\theta^2}$$

On average, the kinetic and potential energies are equal so if θ_0 is the angle at the maximum of the swing:

$$\begin{aligned}\bar{E} =& \frac{mg\ell}{2}\theta_0^2 \\ \Rightarrow Q =& \frac{\bar{E}}{\bar{\Delta E}} \\ =& \frac{\omega}{\bar{p}}\frac{mg\ell}{2}\theta_0^2 \\ =& \frac{\omega T}{MgL}\frac{mg\ell}{2}\theta_0^2\end{aligned}$$

We can now plug in the numbers the problem gives us to find:

$$Q = 68$$

The amount of energy required to drive the clock for one day is:

$$\begin{aligned}E =& MgL \\ =& 0.2\text{kg} \cdot g \cdot 2\text{m} \\ =& 4\text{J}\end{aligned}$$

If the battery only holds 1 joule of power, then it will only make it through a quarter of the day (6 hours).

K + K 10.12

The equation of motion for the free damped oscillator is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

The equation of motion for the forced damped oscillator is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

Let $x_a(t)$ be a solution for the first equation and let $x_b(t)$ be a solution for the second. We also say $x_c(t) = x_a(t) + x_b(t)$. Then we substitute $x_c(t)$ into the second DE.

$$\begin{aligned}\ddot{x}_c + \gamma\dot{x}_c + \omega_0^2 x_c =& \ddot{x}_a + \gamma\dot{x}_a + \omega_0^2 x_a + \ddot{x}_b + \gamma\dot{x}_b + \omega_0^2 x_b \\ =& \frac{F_0}{m}\cos(\omega t)\end{aligned}$$

So we see that $x_c(t)$ satisfies this differential equation.
Now we see that,

$$x_a(t) = A\cos(\omega t + \varphi)$$

And that

$$x_b(t) = Be^{-\gamma t}\cos(\omega_1 t) + Ce^{-\gamma t}\sin(\omega_1 t)$$

Where $\omega_1^2 = \omega_0^2 - \gamma^2/4$. Now the initial condition requires that

$$x_c(0) = 0 \; \dot{x}_c(0) = 0$$

The conditions require that:

$$\begin{aligned}A\cos\varphi + B =& 0 \\ -A\omega\sin\phi - B\gamma + C\omega_1 =& 0\end{aligned}$$

At resonance, $\omega = \omega_0$. Also, $A = \frac{F}{m\omega_0\gamma}$. $\varphi = \arctan\infty = \frac{\pi}{2}$. Then:

$$\begin{aligned}B =& 0 \\ C =& \frac{A\omega}{\omega_1} = \frac{F_0}{m\omega_1\gamma} \\ \gamma \ll 2\omega_0 \Rightarrow C \approx& \frac{F}{m\omega_0\gamma}\end{aligned}$$