Physics 21 : Week 1

Welcome!

I hope everyone has taken Physics 20. If not, please come see me.

Course organization.

Syllabus We're going to be following K & K

Chap 6 & 7, parts of 8 + 9 (small)

Chap 10

Big themes : angular momentum, conservation laws.

Simple harmonic oscillator ← arises in many fields of physics, from condensed matter quantum mechanics early universe/inflation

You'll be seeing these concepts again when you learn quantum mechanics → spin, orbital angular momentum SHO.

Don't worry if you don't get things at ~~first~~ :

"Learning physics is like painting a house" – R. Feynman [you have to apply many coats].

Less exalted version : "You don't understand things, you just get used to them."

Main thing : – do all the problem sets, work lots of problems.

or playing basketball

It's like getting to Carnegie Hall: no use just understanding lectures / theory, you have to practice, practice, practice!

~~Go to~~

Problem sets : – due Mondays 5 p.m., in boxes outside PCS.
sometimes Tues.

Get help early : – go to recitations – (on Fridays). → 11, 1, 2, 3.
– go to office hours

Office hours : Mine : Fri 2-5 p.m., Broida 2015J.

TA Gavin Hartnett ~~Thurs~~ Mon 11:30-2:30 PSR

TA Katherine Hyatt. Thurs 11-12, 2-4 PSR

Grading HW : 30%
Midterm (Feb 11) 25%
Final – 45%.

Website http://web.physics.ucsb.edu/~phys21

→ has HW, solutions
lecture schedule
class announcements → check carefully
(change to PS, etc).

Angular Momentum

This week: K&K 6.1–6.3

Analog of linear momentum (conserved in systems w/ translational symmetry)

Angular momentum is consered in systems w/ rotational symmetry.

Torque → analog of force (give rate of change of linear momentum)

~~6.1 Introduction~~

∴ Torque give rate of change of angular momentum.

6.1 INTRODUCTION

We want to understand the motion of ~~any~~ a rigid body ~~a~~ under <u>any</u> combination of applied forces.

Chasle's theorem: Any displacement of a rigid body can be decomposed into 2 independent motions:

① ~~translatio~~ motion of center of mass ← you know how to do this

② rotation about center of mass ← this bit is new

[SHOW THIS FOR A BOOK] → proven at end of chapter.

This decomposition is <u>not</u> unique, but it always exists.

It is ~~a~~ very useful ~~way of thinp thinkip about thngs.~~

note: we don't <u>need</u> this machinary. They all obey Newton's law, which you already know. It just makes life easier

We can use previous concepts fr. translational motion

translational		rotational
"resistance to motion" ~~center~~ mass	⟷	moment of inertia ← "resistance to rotation"
momentum	⟷	angular momentum
force	⟷	torque.

$$F = \frac{dp}{dt} \qquad\qquad \tau = \frac{dL}{dt}.$$

~~We can consider translational motion in terms of the~~

Let us begin by considering a point particle, with no internal structure ~~Althou~~ → thus, it only exhibits translational motion.

just look at this

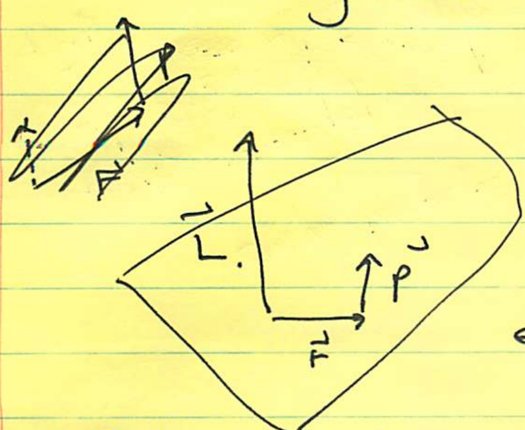
We analyse this in terms of the concepts of angular momentum & torque.

A ~~extended~~ rigid body is simply an ensemble of such particles.

6.2. ANGULAR MOMENTUM

$$\vec{L} = \vec{r} \times \underbrace{\vec{p}}_{[m v]}$$

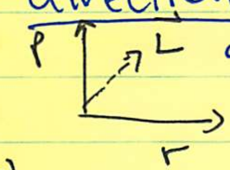
$$[L] = \text{cm}\,[\text{kg cm s}^{-1}]$$
$$= \text{kg cm}^2\,\text{s}^{-1}.$$



Direction of $\vec{L}$ is orthogonal to $\vec{r}$ & $\vec{p}$ (which define a plane).

e.g. if $\vec{r}$ in $\hat{x}$ direction
$\vec{p}$ in $\hat{y}$ direction.
$\vec{L} = \vec{r} \times \vec{p}$ in $\hat{z}$ direction

Only direction and magnitude of $\vec{L}$ matter (not origin)



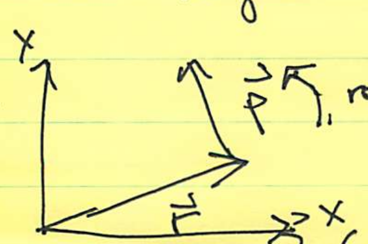
Follows the "right hand rule" for cross products

DIRECTION

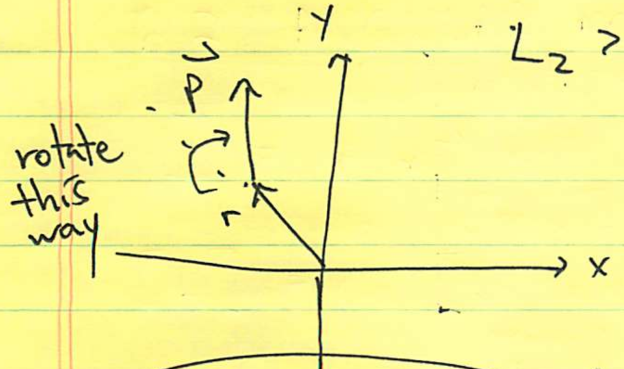
↳ ① point right palm in $\vec{r}$ direction, sweep out to $\vec{p}$ direction – thumb gives $\vec{L}$ direction.

② $\vec{r}$ = thumb
$\vec{p}$ = 2nd finger
$\vec{L}$ = 3rd finger.

Practice:



$L_z > 0$ or $L_z < 0$? (ans: > 0).



$L_z > 0$ or $L_z < 0$?

ans ($L_z < 0$.)

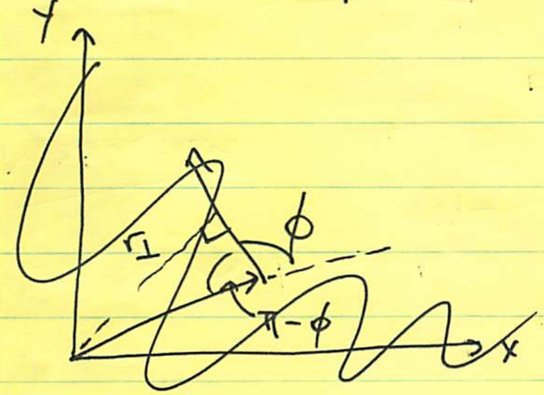
MAGNITUDE

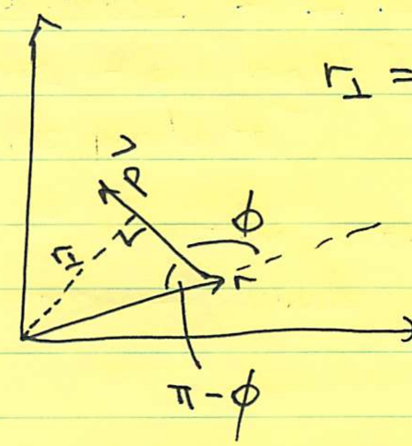
Book presents 3 ways of lookip at it

just ~~review~~

Method 1:

$L = \vec{r} \times \vec{p} = rp\sin\phi$

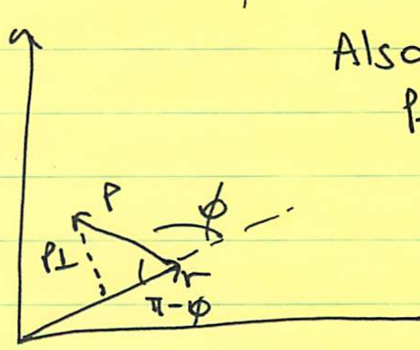




$r_\perp = r\sin(\pi-\phi)$
$= r\sin\phi$

$L_z = r_\perp p$

~~Note~~



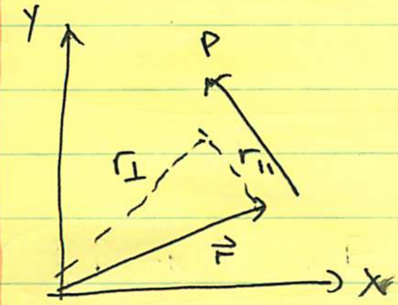
Also
$p_\perp = p\sin(\pi-\phi)$
$= p\sin\phi$

$L_z = r p_\perp$

Note that $|\vec{L}|$ depends on choice of origin.

Lect 1 → 236.
2 → 241
3 → 247.

Method 2



Resolve $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$

$\parallel$ & $\perp$ are relative to $\vec{p}$.

$$\vec{L} = \vec{r} \times \vec{p}$$
$$= (\vec{r}_{\perp} + \vec{r}_{\parallel}) \times \vec{p}$$
$$= (\vec{r}_{\perp} \times \vec{p}) + \cancel{(\vec{r}_{\parallel} \times \vec{p})} \rightarrow 0$$

Since these vectors are $\perp$,

$$|\vec{L}| = |\vec{r}_{\perp} \times \vec{p}| = |r_{\perp}|\,|p|.$$

Similarly, set

$$\vec{p} = \vec{p}_{\parallel} + \vec{p}_{\perp},$$

get

$$|L| = |r|\,|p_{\perp}|$$

Method 3

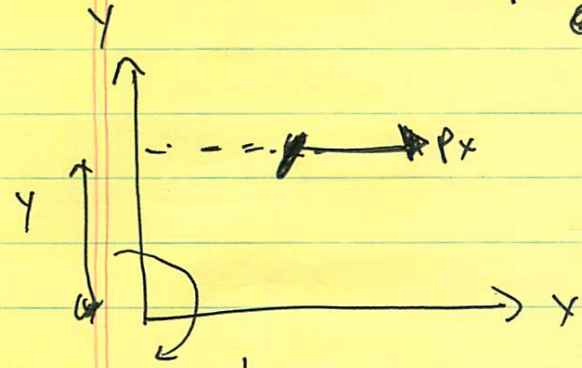
Consider motion in x-y plane, so that

$$\vec{r} = (x, y, 0)$$
$$\vec{p} = (p_x, p_y, 0).$$

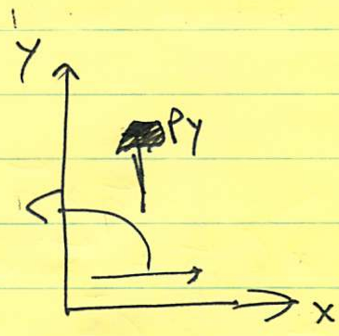
Then $L = \vec{r} \times \vec{p}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix} = (x p_y - y p_x)\hat{k}$$

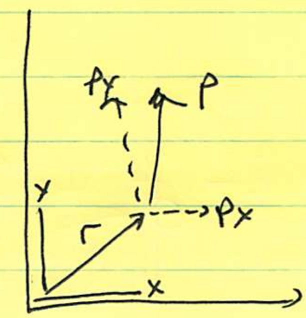
We can understand this ~~graphically~~
from the same as before:
parallel components don't contribute to cross product, only $\perp$ cpt



$L_z = -y p_x$



$L_z = x p_y$



$L_z = x p_y - y p_x$.

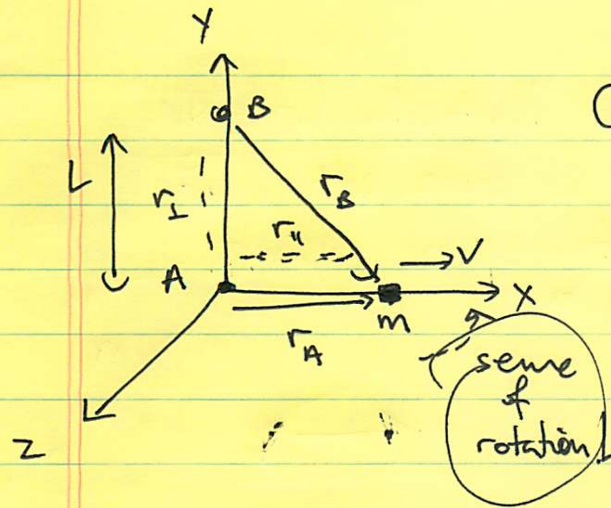
can decompose
$$r = x\hat{x} + y\hat{y}$$
$$p = p_x\hat{x} + p_y\hat{y}$$

Can do this more generally when motion is not in xy plane:

$$\vec{L} = \vec{r} \times \vec{p}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (\quad)\hat{i} + (\quad)\hat{j} + (\quad)\hat{k}.$$

Examples $\rightarrow$ show how $\vec{L}$ depends on choice of origin.

6.1 Angular Momentum of a Sliding Block



Consider how $\vec{L}$ changes if origin is at A or at B.

From A: $\vec{p} = m\vec{v}$

$\vec{L}_A = \vec{r}_A \times \vec{p} = 0.$

since r_A $\parallel$ to $\vec{v}$.

From B:

$$L_B = \vec{r}_B \times \vec{p} = (\vec{r}_\parallel + \vec{r}_\perp) \times m\vec{v}$$

$$= r_\perp \times m\vec{v}$$
$$= mlv\,\hat{k}$$

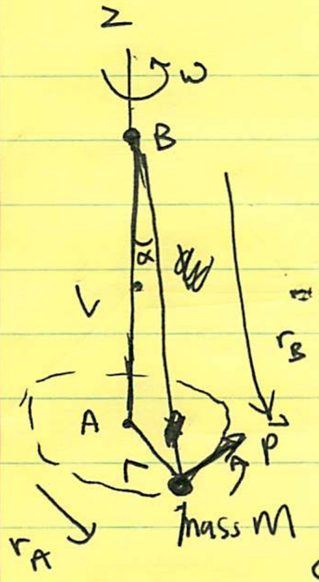
or, can write

$$r_B = x\,\hat{i} - l\,\hat{j}$$

$$\vec{L}_B = \vec{r}_B \times m\vec{v}$$
$$= m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & -l & 0 \\ v & 0 & 0 \end{vmatrix} = mlv\,\hat{k}$$

<u>Example 6.2</u>

Angular Momentum of Conical Pendulum.



First consider $\vec{L}_A$ (origin is A).

$$\vec{L}_A = \vec{r}_A \times \vec{p} = rp\,\hat{k}$$

$$|p| = Mv = Mr\omega$$

$$\Rightarrow \vec{L}_A = Mr^2\omega\,\hat{k}$$

constant in both magnitude and direction.

Consider $\vec{L}_B$.

First consider magnitude

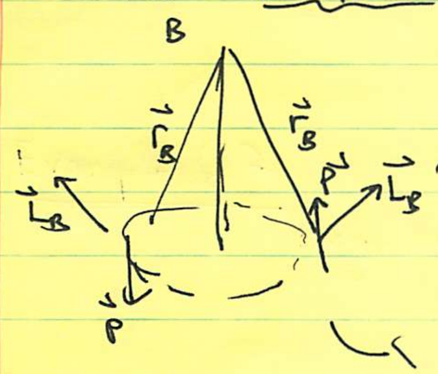
$$|\vec{L}_B| = |\vec{r}_B \times \vec{p}| = |\vec{r}_B|\,|p|$$ since $\vec{r}_B$ & $\vec{p}$ are $\perp$

$$= lp = lMr\omega$$

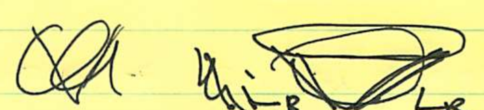
which is different from $|\vec{L}_A|$ → so $|\vec{L}|$ depends on the origin we choose.

Now consider direction.

While $|r|$, $|p|$ are const with time, the direction of $\vec{r}$, $\vec{p}$ constantly change w/ time



→ $\vec{L}_B$ forms a cone.



(z component is const., but horizontal cpt travels around circle w/ bob).

6-3 TORQUE

Consider Newton's 2^{nd} law:

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

Let's consider an analogous quantity, torque

$$\tau = \frac{d\vec{L}}{dt}$$

$$= \frac{d}{dt}(\vec{r} \times p)$$

$$= \left(\frac{d\vec{r}}{dt} \times p\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

$$= \cancel{(\vec{v} \times m\vec{v})}^{0} + (\vec{r} \times \vec{F})$$

$$\Rightarrow \boxed{\tau = \cancel{\vec{r} \times \vec{p}}\ \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}$$

So if $\tau = 0 \Rightarrow \vec{L} = \text{const}$

angular momentum is conserved if no torques act on system

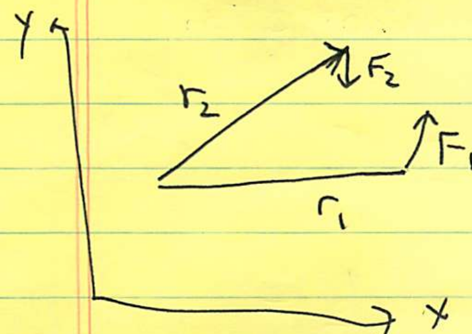
(Just as no forces $\rightarrow$ linear momentum is conserved we'll see this more later).

As before, $|\tau| = |r_\perp||F| = |r||F_\perp|$

or

$$\tau = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

Can also associate direction w/ torque



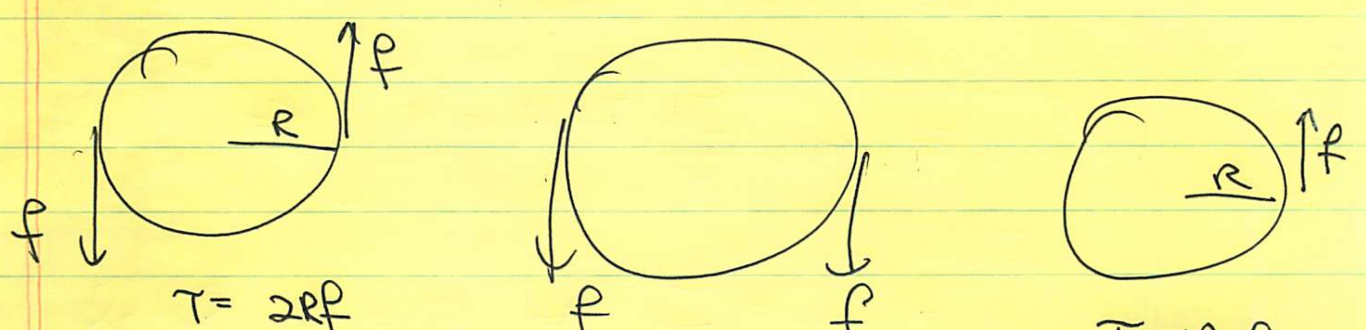
$(\tau_2 < 0)$

$(\tau_2 > 0)$.

Torque differs from force:

1) $\vec{\tau}$ depends on origin, $\vec{F}$ does not.

2) $\vec{\tau} = \vec{r} \times \vec{F} \rightarrow \vec{\tau}$ & $\vec{F}$ always $\perp$.



$\tau = 2Rf$

$F = 0$

torque w/out force

$\tau = 0$

$F = 2f$

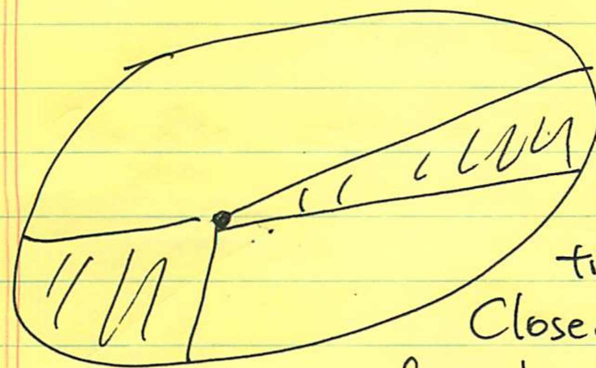
Force w/out torque

$\tau = Rf$

$F = f$.

In general, always have torque + force.

Example 6.3 Central Force Motion, law of equal areas



1609 Kepler:
2nd law of planetary motion: equal areas swept out in equal time.
Closer to Sun - move faster - compensate for shorter radius.

we show
→ Comes fr. conservation of angular momentum
→ holds not just for gravity, but for any central force.

First show $\vec{L}$ is conserved:

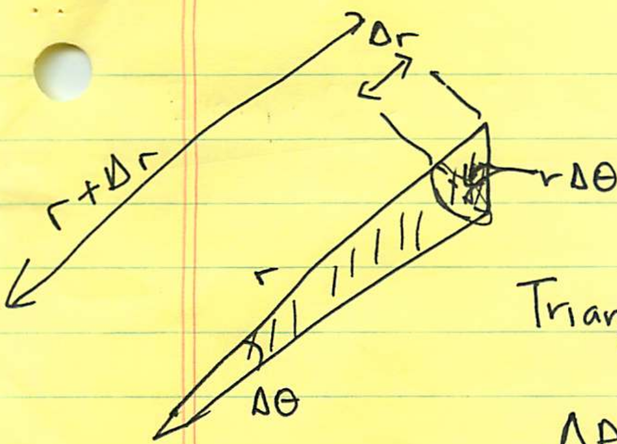
$\vec{F} = f(r)\,\hat{r}$ (central force, not just gravity; for gravity, $f(r) = \frac{GM_1m_2}{r^2}$)

Torque $\tau = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = 0$

$\Rightarrow \vec{L}$ is conserved.

$\Rightarrow$ motion must lie in a plane (otherwise direction of $\vec{L}$ would change)

Now show $\frac{dA}{dt}$ is constant



t $\qquad$ $t + \Delta t$

(r, θ) $\qquad$ $(r+\Delta r, \theta + \Delta\theta)$

Triangle's area swept out:

$$\Delta A = \tfrac{1}{2}\,\text{base} \times \text{height}$$

$$= \frac{1}{2}(r+\Delta r)(r\Delta\theta)$$

$$= \frac{1}{2} r^2 \Delta\theta + r\Delta\theta\Delta r \simeq \frac{1}{2} r^2 \Delta\theta$$

this is 2nd order quantity

Area is swept out at rate:

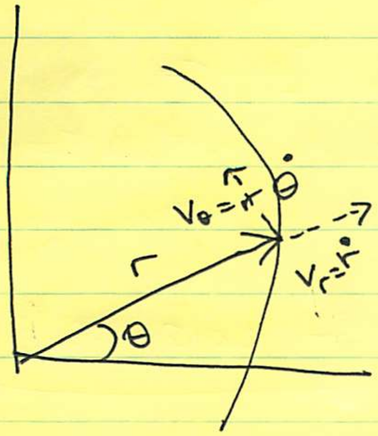
$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t}$$

contribution of large triangle

contribution of small triangle

$$= \lim_{\Delta t \to 0} \frac{1}{2}\left(r^2 \frac{\Delta\theta}{\Delta t} + \cancel{r\frac{\Delta\theta\Delta r}{\Delta t}}\right)$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt}$$



$$\vec{v} = v_r \hat{r} + v_\theta \hat{\theta}$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

So $\vec{L} = \vec{r} \times \vec{p}$

$$= \vec{r} \times m\vec{v}$$

$$= r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$\Rightarrow \vec{L} = m r^2 \dot{\theta}\, \hat{k} \qquad (\hat{r} \times \hat{\theta} = \hat{k}).$$

(since 2 $\perp$ vectors in a plane).

So $$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_z}{2m} = \underline{\underline{\text{const}}}$$

as in Kepler's law.

~~Another way:~~
~~for central force, $F_\theta = 0$~~
~~acceleration $\Rightarrow a_\theta = 0$~~

second way is harder

$\to$ requires $\frac{d\hat{r}}{dt}$, $\frac{d\hat{\theta}}{dt}$.

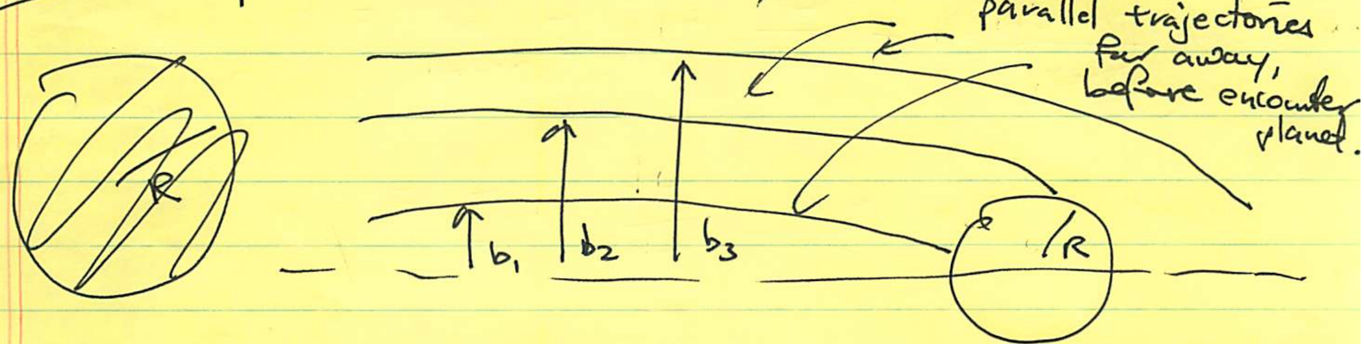
~~$a_\theta = \frac{d}{dt}(v_\theta) = \dot{r}\dot{\theta} + r\ddot{\theta} = 0$~~

Take $\frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$

~~$\frac{d}{dt}(r\dot{\theta}) = \dot{r}\dot{\theta} + r\ddot{\theta}$~~

~~$\Rightarrow r a_\theta = r\dot{r}\dot{\theta} + r^2\ddot{\theta}$~~

Example 6.4 Capture cross-section of a Planet.



Consider a planet of radius $R \to$ area of disk is $A = \pi R^2$.

~~Thi This is~~

However, gravity increases the effective cross-section which are aimed outside the disk
$\to$ some trajectories are deflected toward planet, and still end up hitting it.

Complicated way : use Newton's laws of motion, work out orbit in grav. field of planet.

Easy way : use conservation of energy & angular momentum.

Shows power of conservation laws!

Let's find the largest value of impact parameter b', where will still strike planet.

Ask

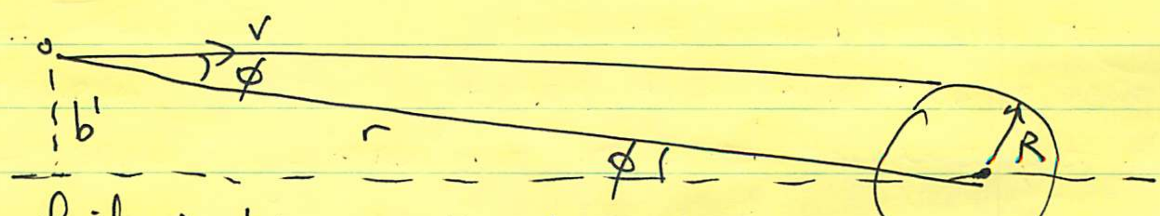
N.B. linear momentum of spacecraft is not conserved since there is a gravitational force ~~acted~~ acting on it. The combined linear momentum of spacecraft + planet is conserved [so, center of mass of system obeys conservation of linear momentum].

Energy conservation

$$E = K + U$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Angular momentum conservation



impact parameter for trajectory which just grazes planet

Angular momentum about center of planet

$$L = \vec{r} \times \vec{p}$$
$$= -rp\sin\phi$$
$$= -m r v \sin\phi$$

Initially $r \to \infty$

$$v = v_0$$
$$r\sin\phi = b'$$

$$\Rightarrow L = -mb'v_0$$
$$E = \frac{1}{2} m v_0^2 .$$

At point of closest approach, $r = R$ (since this trajectory "just grazes" the planet).

At this point $r \perp v \Rightarrow \sin\phi = 1$.

$$L = -mR\underbrace{v(R)}_{\text{speed at point of closest approach}}$$

$$E = \frac{1}{2} m v(R)^2 - \frac{GMm}{R}$$

Since L & E are conserved, we can equate these to the initial values:

$$-mb'v_0 = -mRv(R) \quad (1)$$
$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v(R)^2 - \frac{GMm}{R} \quad (2)$$

Fr. equation (1),

$$v(R) = v_0 \frac{b'}{R} \qquad \left(\text{Note } \frac{b'}{R} > 1\right)$$

Fr. equation 2

$$v(R)^2 - v_0^2 = \frac{2GM}{R}$$

$$\Rightarrow v_0^2 \left(\frac{b'^2}{R^2} - 1\right) = \frac{2GM}{R}$$

$$\Rightarrow b'^2 = R^2 \left(\frac{2GM}{Rv_0^2} + 1\right)$$

Thus, the effective area is

$$A_e = \pi (b')^2$$

$$= \pi R^2 \left(1 + \frac{2GM}{Rv_0^2}\right)$$

$$= \underbrace{\pi R^2}_{\text{geometrical area } A_g} \left(1 + \frac{\overbrace{\frac{GMm}{R}}^{= U(R)}}{\underbrace{\frac{1}{2}mv_0^2}_{E}}\right)$$

$$= A_g \left(1 - \frac{U(R)}{E}\right) > A_g \quad \left(\begin{array}{c}\text{since}\\ U(R) < 0\end{array}\right)$$

Suppose we "turn off" gravity, $G \to 0$

Then $U(R) \to 0$, $A_e \to A_g$ ✓

Suppose you start from rest $\Rightarrow v_0 = 0$

$$\Rightarrow E = \frac{1}{2} m v_0^2 = 0.$$

Then

$$A_e = A_y\left(1 - \frac{U_r}{E}\right) \rightarrow \infty \Rightarrow \text{impossible to miss planet!}$$

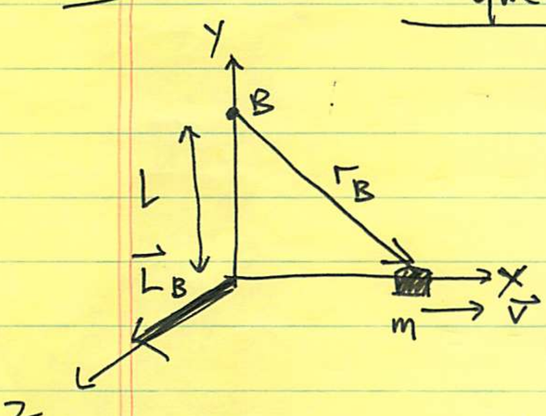
(makes sense, since all forces are directed toward planet. The initial velocity provides momentum which enables you to "resist" the planet.

Ex

$$\boxed{\text{Examples of } \frac{dL}{dt} = \tau}$$

Example 6.5

Torque on a Sliding Block.



Consider a small block of mass m sliding in the x direction

$$\vec{v} = v\hat{x}.$$

Angular momentum about origin B:

$$L = \vec{r} \times \vec{p}$$
$$= m\vec{r}_B \times \vec{v}$$
$$= mlv\,\hat{z}$$

If block is sliding freely, ~~there is~~ $v = \text{const}$

$$\Rightarrow \vec{L} = \text{const}.$$

Suppose block slows due ~~due~~ to frictional force:

$$\vec{F} = -f\hat{x}$$

Then

$$\tau_B = \vec{r}_B \times f$$
$$= -lf\ \hat{z}$$

The frictional force slows down the block and reduces its angular momentum

$$L = mlv\,\hat{z} \Rightarrow \Delta L = ml\,\Delta v\,\hat{z}$$

where $\Delta v < 0$

change in magnitude, but not direction.

$$\frac{dL}{dt} = \Rightarrow \frac{\Delta L}{\Delta t} = mL\,\frac{\Delta v}{\Delta t}\,\hat{z}$$
$$= -lf\ \hat{z}$$
$$= \tau_B \quad \checkmark$$

Note: Since τ, L depend on choice of origin, must be careful to use same origin for both.

This example: const direction, change in magnitude

Now consider case when $\vec{L}$ ~~changes direction~~. has const magnitude, but changes direction.

Example 6.6

Torque on Conical Pendulum

Before we showed that $\vec{L}_A \neq \vec{L}_B$ for a conical pendulum, where A & B are different choices for the origin.

Now show that $\vec{\tau} = \frac{d\vec{L}}{dt}$ is satisfied in this situation.

Consider origin A

There is no vertical acceleration:

(1) $T\cos\alpha = Mg$ [ie, no net vertical force]

Total force on ~~the~~ bob is radially inward:

$$\vec{F} = -T\sin\alpha\,\hat{r}$$

$$\tau_A = \vec{r}_A \times \vec{F} = 0$$

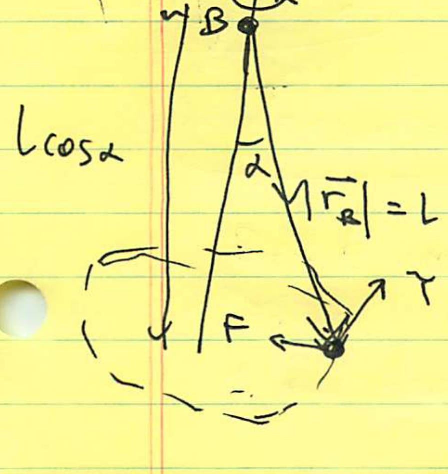
$$\Rightarrow \frac{dL}{dt} = \tau = 0 \Rightarrow L_A = \text{const.}$$

(as we already showed)

Consider origin B (remember here $\vec{L}$ kept changing direction).

$$\vec{\tau}_B = \vec{r}_B \times \vec{F}.$$

Hence:



$$|\tau_B| = L\sin(\pi - \alpha)\,F$$
$$= L\cos\alpha\,F$$
$$= L\cos\alpha\,T\sin\alpha$$
$$= MgL\sin\alpha$$

(from eqn of vertical balance)

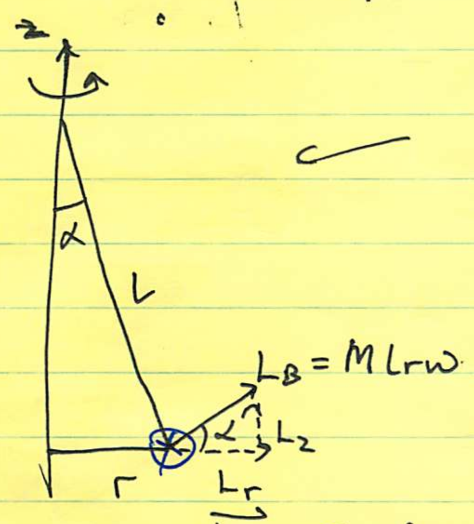
τ_B is tangential to line of motion of m

$$\tau_B = Mgl \sin\alpha \, \hat{\Theta}$$

Need to show that $\vec{\tau}_B = \frac{d\vec{L}_B}{dt}$ is satisfied.

Fr. Example 6.2, we know that

$$|\vec{L}_B| = Mlr\omega = \text{const}.$$



Split $\vec{L}$ into

$$\vec{L} = \vec{L}_z + \vec{L}_r$$

where

$$|L_z| = Mlr\omega \sin\alpha$$

$$|L_r| = MLr\omega \cos\alpha.$$

$\vec{L}_z$ is const (as expected, since $\vec{\tau}$ has no $\vec{z}$ cmpt).

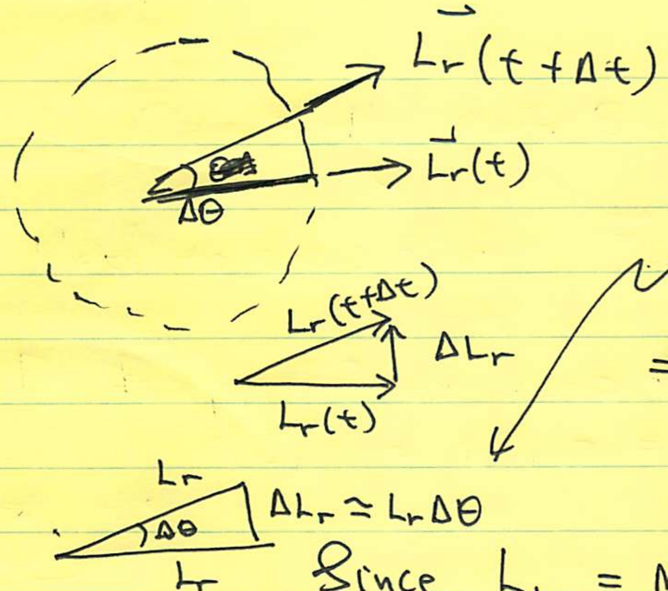
While $|L_r| = \text{const}$, $\vec{L}_r$ is time-dependent (continuously change direction).

<u>Section 1.8</u> → showed that the only way a vector $\vec{A}$ of const magnitude can change is to rotate

$$\left|\frac{d\vec{A}}{dt}\right| = |A| \frac{d\theta}{dt}.$$

$$\Rightarrow \left|\frac{d\vec{L}_r}{dt}\right| = |\vec{L}_r|\omega$$

Let's try to derive this geometrically.



$$\Delta L_r \simeq L_r \Delta\theta$$

$$\Rightarrow \frac{dL_r}{dt} = L_r \frac{d\theta}{dt}$$

$$= L_r \omega .$$

Since $L_r = MLr\omega\cos\alpha$

$$\Rightarrow \quad \frac{dL_r}{dt} = MLr\omega^2\cos\alpha$$

$$= \left[Mr\omega^2\right] L\cos\alpha$$

$T\sin\alpha$ (since this is the radial force

$$= T\sin\alpha \, L\cos\alpha$$

$$= (T\cos\alpha) \, L\sin\alpha$$

$$= MgL\sin\alpha$$

$$= \tau_B \checkmark$$

Also, $\frac{d\vec{L}_r}{dt}$ is in tangential direction, ‖el to $\vec{\tau}_B$.

Another way: — just write $\vec{L}_B$ explicitly as a vector, and differentiate it.

$$\vec{L}_B = \underbrace{(M l r \omega \sin\alpha)}_{\text{const}}\,\hat{z} + \underset{\uparrow\ \text{const.}}{(M l r \omega \cos\alpha)}\,\hat{r}$$

$$\frac{d\vec{L}_B}{dt} = M l r \omega \cos\alpha \,\frac{d\hat{r}}{dt}$$

But this is a rotating vector $\left|\frac{d\hat{r}}{dt}\right| = \omega|\hat{r}| = \omega$

~~and~~ $\rightarrow \frac{d\hat{r}}{dt} = \omega\,\hat{\theta}$

$$\Rightarrow \frac{d\vec{L}_B}{dt} = M l r \omega^2 \cos\alpha\,\hat{\theta}$$

Important to be able to visualize $\vec{L}$ as a vector which rotates in space!

We will use this to analyze gyroscopes later.

Example 6.7

Torque due to Gravity

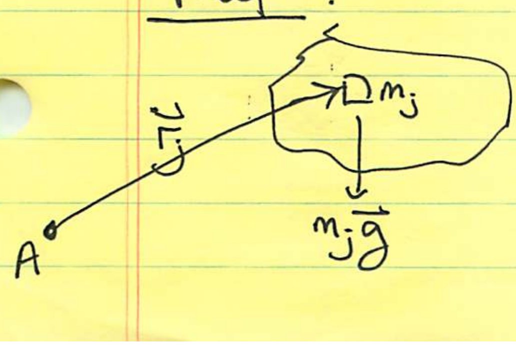
Force due to gravity can exert torque (e.g., pendulum, top).

Statement For a uniform gravitational field,

$$\vec{\tau}_{grav} = \vec{r} \times \vec{w} \leftarrow \text{weight}.$$

$\uparrow$ vector fr. origin to center of mass

Proof :



Regard the body as a collection of particles.

The mass element m_j experiences a torque

$$\vec{\tau}_j = \vec{r}_j \times m_j \vec{g}$$

Total torque:

$$\tau = \sum \tau_j = \sum \vec{r}_j \times m_j \vec{g}$$

$$= \sum m_j \vec{r}_j \times \vec{g}$$

But $\sum m_j \vec{r}_j = M\vec{R}$ ← position vector of center of mass.

$$\Rightarrow \tau = M\vec{R} \times \vec{g}$$

$$= \vec{R} \times M\vec{g}$$

$$= \vec{R} \times \vec{W} \quad \checkmark$$

Thus for an object to be balanced, the pivot point must be at the center of mass → otherwise, it will experience a torque due to gravity.