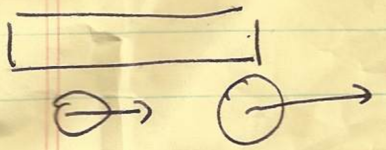
Week 2

Angular Momentum and Fixed Axis Rotation

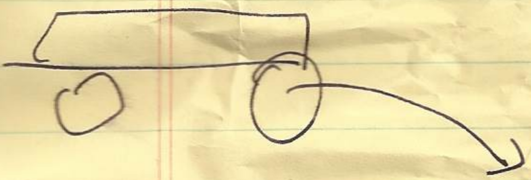
Consider motion of rigid bodies (→ shape doesn't change.) → most general case is application of Chasle's thm (translation + rotation about a fixed axis → e.g.

baseball bat spinning + tumbling through air.

We consider rotation about a fixed axis
→ axis can translate, but points in a fixed direction.

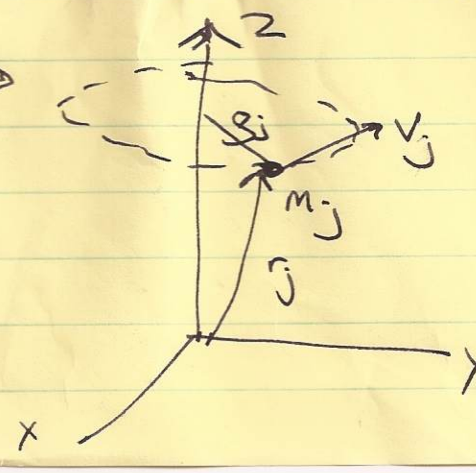


car going straight
→ fixed axis rotation



car turns
→ direction of axis changes
→ no longer fixed axis rotation.

Choose axis in $\hat{z}$ direction.
Rigid body → $|r|$ = const for each particle
→ ~~circular motion~~ ($v \perp r$ always, $\dot{r} = 0$)



$|v_j| = \rho_j \omega$ ← angular velocity

ρ_j: ⊥ distance fr. axis of rotation (r is distance fr. origin)

$$\rho_j = (x_j^2 + y_j^2)^{1/2}$$
$$r_j = (x_j^2 + y_j^2 + z_j^2)^{1/2}$$

Angular momentum of j^{th} particle is
$$L(j) = \vec{r}_j \times m_j \vec{v}_j$$

Here we focus on $L_z \rightarrow$ component of $\vec{L}$ along axis of rotation.

Since $\vec{v}_j \perp$ to z-axis, it lies in the x-y plane.
Thus,
$$L_z(j) = m_j v_j \times (\text{dist. to z axis}) = m_j v_j \rho_j$$
$$= m_j \rho_j^2 \omega$$

Sum over all particles
$$L_z = \sum L_z(j)$$
$$= \left[\sum_j m_j \rho_j^2\right] \omega$$

note that ω is constant for all particles $\rightarrow$ otherwise body will deform

MOMENT OF INERTIA

Can ~~write this as~~ ~~$L_z = I\omega$~~

spokes will deform.

$$L_z = I\omega \quad \text{where } I = \sum_j m_j \rho_j^2$$

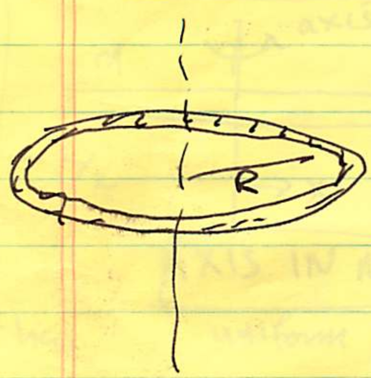
Analogy of mass for rotation: depends on distribution of mass axis of rotation.

For continously distributed matter,

$$I = \sum_j m_j \rho_j^2 \rightarrow \int \rho^2 \, dm$$
$$= \int (x^2 + y^2) \, dm$$
$$= \int (x^2 + x^2) \underset{\substack{\uparrow \\ \text{density}}}{\omega} \, dV$$

(we use ω for density, since we use ρ for $\perp$ dist).

MOMENT OF INERTIA for some simple objects



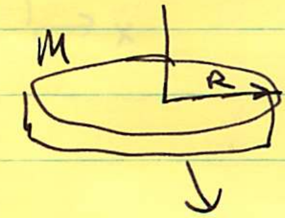
$\boxed{\text{Thin hoop of mass } M \text{, radius } R}$

$$\cancel{dm} \; I = \int \rho^2 \, dm$$
$$= \int_0^{2\pi R} \rho^2 \lambda \, ds \quad \text{where}$$

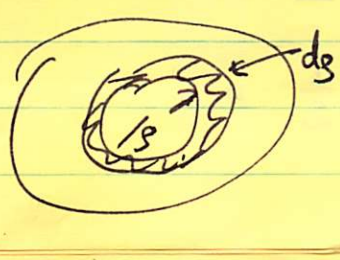
$\lambda = \frac{M}{2\pi R}$ is mass per unit length

$$= \int_0^{2\pi R} R^2 \left(\frac{M}{2\pi R}\right) ds$$
$$= \frac{MR}{2\pi} \cdot 2\pi R = MR^2$$

$\boxed{\text{Uniform Disk}}$



We can divide into series of thin hoops
w/ radius ρ
width $d\rho$
moment of inertia dI.



For continously distributed matter,

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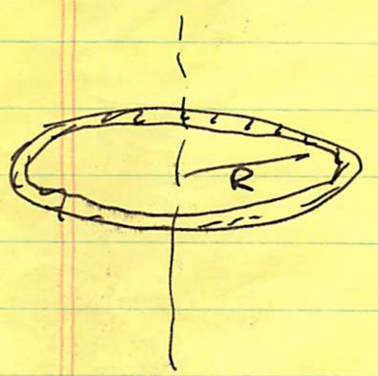
$$= \int (x^2 + x^2) \, \omega \, dV$$

density

(we use ω for density, since we use ρ for $\perp$ dist)

MOMENT OF INERTIA for some simple objects

$\boxed{\text{Thin hoop of mass } M\text{, radius } R}$



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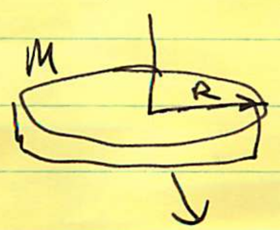


We can divide into series of thin hoops w/ radius ρ width $d\rho$ moment of inertia dI.

$$I = \int \rho^2 \, dm$$

Let's first consider the mass of a thin hoop

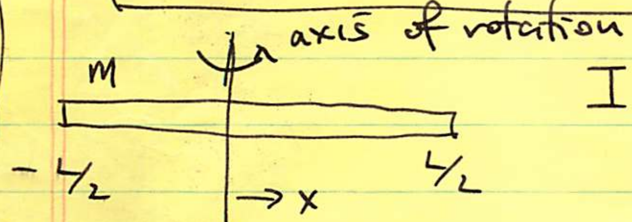
$$dm = M\left(\frac{dA}{A}\right) = \frac{M(2\pi \rho \, d\rho)}{\pi R^2} = \frac{2M\rho \, d\rho}{R^2}.$$

$$dI = \rho^2 \, dm = \frac{2M\rho^3 d\rho}{R^2}$$

ALWAYS IMPORTANT TO DEFINE axis of rotation

$$I = \int_0^R \frac{2M\rho^3 d\rho}{R^2} = \frac{1}{2} M R^2.$$

Uniform Thin Stick



AXIS IN MIDDLE uniform stick

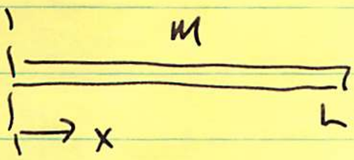
~~e.g.~~ stick has very different moment of inertia if use different axis!

$$I = \int_{-L/2}^{L/2} x^2 \, dm$$

$$= \int_{-L/2}^{L/2} x^2 \left(\frac{M}{L}\right) dx$$

$$= \frac{M}{L} \frac{x^3}{3} \Bigg|_{-L/2}^{L/2}$$

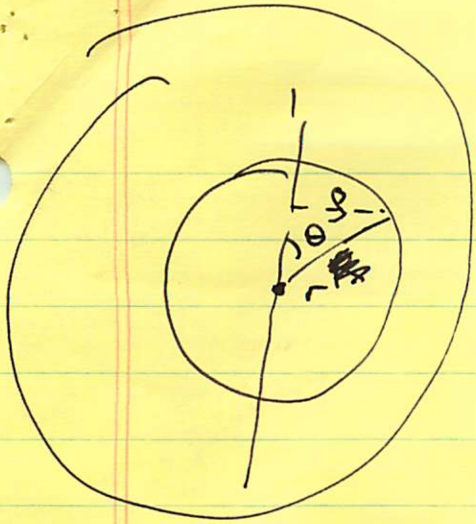
$$= \frac{ML^2}{12}.$$



AXIS AT ONE END uniform stick.

$$I = \frac{M}{L} \int_0^L x^2 \, dx$$

$$= \frac{1}{3} M L^2$$



Uniform sphere of radius R (axis through center).

$$I = \int \rho^2 \, dm \qquad w = \frac{M}{\frac{4}{3}\pi R^3}$$

$$dm = w \, dV$$

$$= w \, 2\pi \, d(\cos\theta) \, r^2 \, dr$$

$$\rho = r \sin\theta$$

$$I = \int \rho^2 \, dm = w \int_{-1}^{1}\int_0^R r^4 \, \cancel{\sin^3\theta \, d\theta} \, \sin^2\theta \, d(\cos\theta) \, dr$$

$$= 2\pi w \int_{-1}^{1} (1 - u^2) \, du \int_0^R r^4 \, dr$$

$$= 2\pi w \left(u - \frac{u^3}{3} \right) \Big|_{-1}^{1} \frac{1}{5} R^5$$

$$= \frac{M}{\frac{4}{3}\pi R^3} \cdot 2\pi \underbrace{\left[\underbrace{\left(1 - \frac{1}{3}\right)}_{2/3} - \underbrace{\left(-1 - \left(-\frac{1}{3}\right)\right)}_{-2/3} \right]}_{4/3} \cdot \frac{1}{5} R^5$$

$$= \frac{2}{5} M R^2 .$$

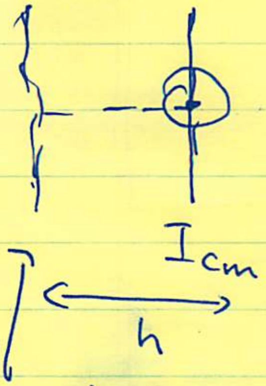
Parallel Axis Theorem

$$I = I_{cm} + Mh^2$$

I_{cm}: moment of inertia about center of mass

M: mass of object

h: perpendicular dist between axes.



moment of inertia about this axis is

$I = I_{cm} + Mh^2$

Note:

1) The axes must be <u>parallel</u>

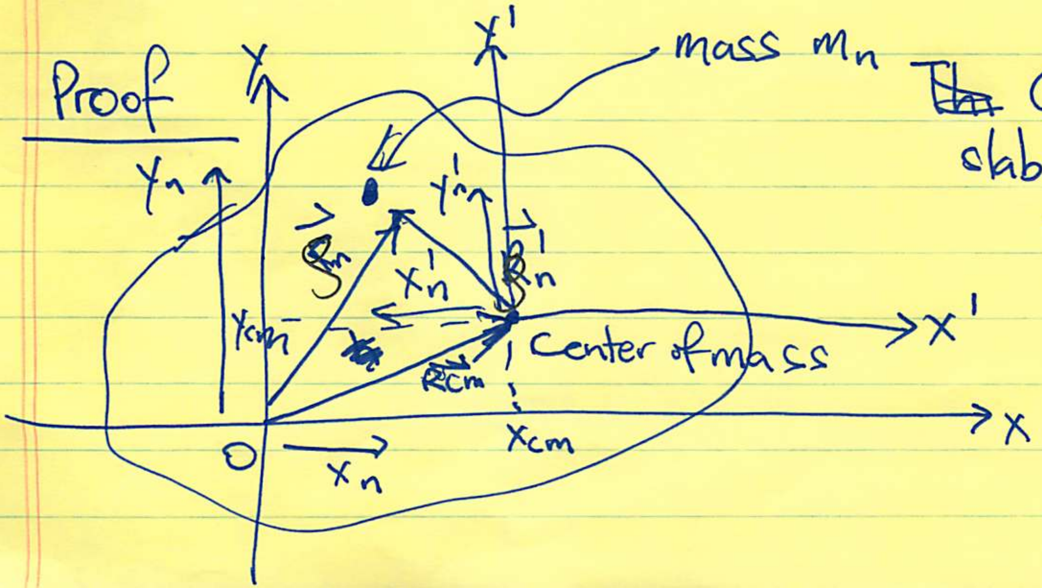
2) $I = I_{cm} + Mh^2 > I_{cm}$

Mh^2 ← this is always +ve

I_{cm} is always the <u>smallest</u> moment of inertia for any set of parallel axes.

(this is <u>not</u> the smallest moment of inertia — an axis pointing in a different direction may have a smaller value).

Proof



~~The~~ Consider thin slab in xy plane to be rotated about z-axis

Then moment of inertia

$$\cancel{I = \sum m_n r_n^2 = \sum m_n (x_n^2 + y_n^2)}$$

$$\cancel{\vec{x}_n = \vec{R}_{cm} +}$$

$$\vec{\rho}_n = \vec{R}_{cm} + \vec{r}_n{}'$$

$$I = \sum m_n \rho_n^2$$

$$= \sum m_n (\vec{R}_{cm} + \rho_n')^2$$

$$= \sum m_n (R_{cm}^2 + 2 R_{cm} \cdot \rho_n' + \rho_n'^2)$$

this term vanishes since it is evaluated in cm system

But note

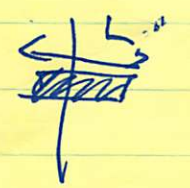
$$\sum m_n \rho_n' = M R_{cm}' = 0$$

$$\Rightarrow \quad I = \sum m_n r_n'^2 + R_{cm}^2 \sum m_n$$

$$= I_{cm} + h^2 M$$

where $|R_{cm}| = h$.

Simple example

Before  $I = \dfrac{ML^2}{12}$

 $I = I_{cm} + m\left(\dfrac{L}{2}\right)^2$

$$I = \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$= \frac{ML^2}{3} \checkmark \quad \text{what we got before.}$$

This makes life very easy!

moment of inertia about axis at rim is:

$$I_z = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

(note: larger).

Dynamics of Pure Rotation about an Axis

Isolated system with no forces acting
⇒ linear momentum is conserved.
by Newton's 3rd law, all internal forces cancel
Similarly: isolated system with no torques acting:
angular momentum conserved.

Let's consider <u>pure rotation</u>, when there is no translation of the axis.
(e.g. → motion of ~~door~~ on hinges
→ spinning of fan blade.

$$L = I\omega$$

$$\Rightarrow \cancel{dt}\ \tau = \frac{dL}{dt} = \frac{d}{dt}(I\omega)$$

$$= I\frac{d}{dt}\omega = I\alpha$$

$\alpha = \frac{d\omega}{dt}$ is <u>angular acceleration</u>

$$\Rightarrow \boxed{\tau = I\alpha}$$

Just like $F = ma$.

Let's keep this analogy going:

$$K = \sum \tfrac{1}{2} m_j v_j^2$$

$$= \sum \tfrac{1}{2} m_j (\rho_j^2 \omega^2)$$

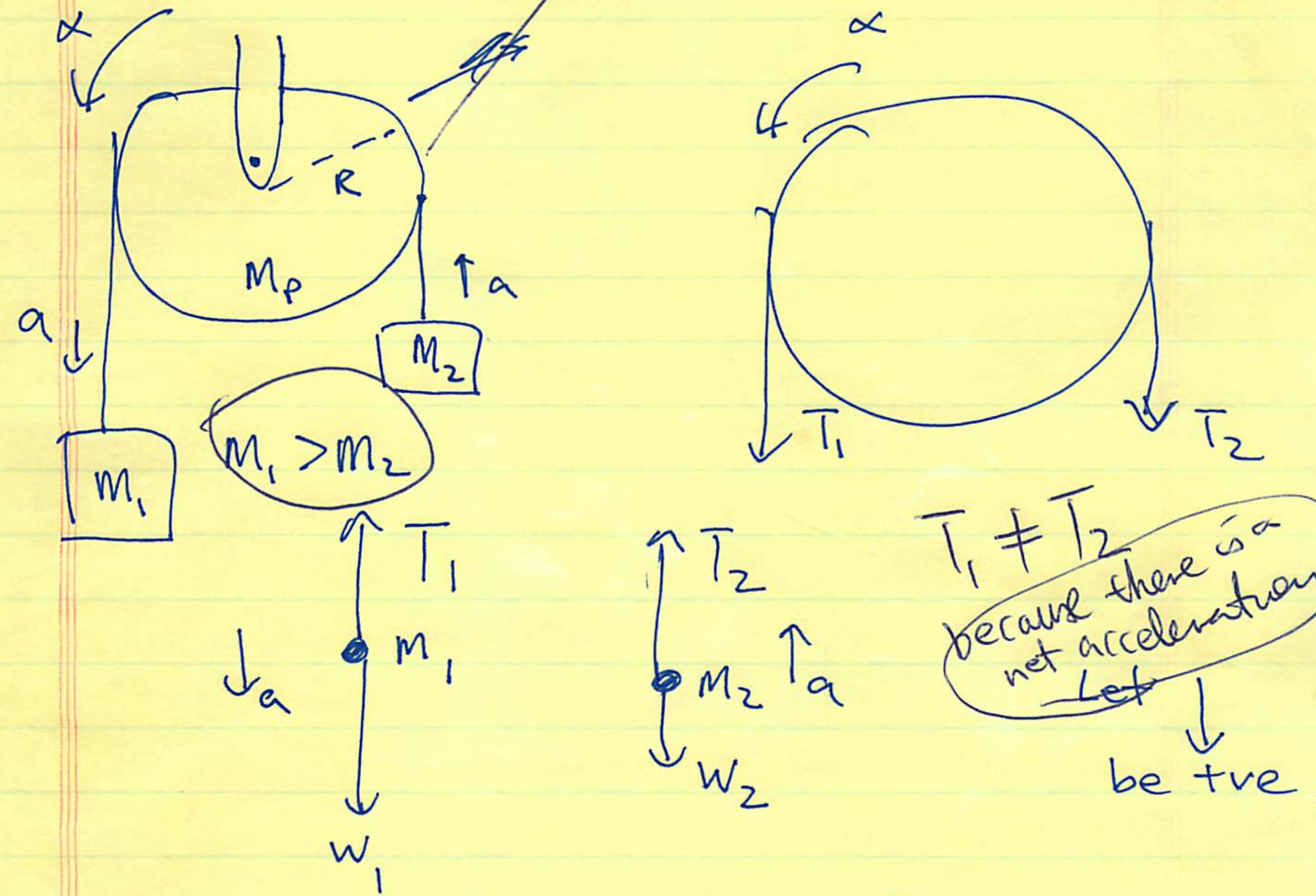
$$= \tfrac{1}{2} I\omega^2$$

Do spinning chair demonstration.

(1a)

Atwood's Machine

First solve system if pulley has ~~no mass~~.



$T_1 \neq T_2$ because there is a net acceleration ~~Let~~ → be tue.

$$W_1 - T_1 = M_1 a \quad (1)$$

$$W_2 - T_2 = -M_2 a \quad (2)$$

change in tension used to apply torque on pulley. → $\tau = T_1 R - T_2 R = I\alpha$ (3)

No slip condition:

$\cancel{a = \alpha R}$

$$V = \omega R$$

$$\Rightarrow \quad a = \alpha R. \quad (4)$$

unknowns:

T_1, T_2, a, α

4 equations,

4 unknowns.

(1) − (2),

$$(W_1 - W_2) - (T_1 - T_2) = (M_1 + M_2)a \qquad (5)$$

From equation (3),

use equation (4)

$$(T_1 - T_2) = \frac{I\alpha}{R} = \frac{Ia}{R^2} \qquad (6)$$

(5) + (6)

$$\Rightarrow \underbrace{(W_1 - W_2)}_{(M_1 - M_2)g} - \frac{Ia}{R^2} = (M_1 + M_2)a$$

$\cancel{\Rightarrow a =}$

$$\Rightarrow a\left(M_1 + M_2 + \frac{I}{R^2}\right) = (M_1 - M_2)g$$

If pulley is a simple disk, $I = \frac{M_p R^2}{2}$.

Then

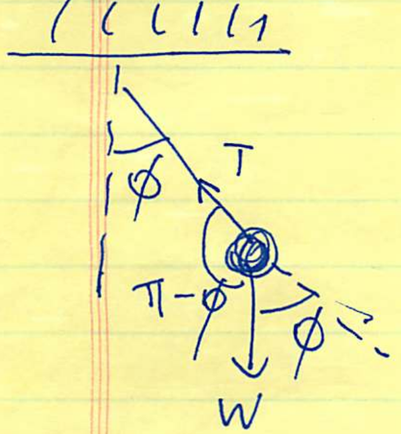
$$a = \frac{(M_1 - M_2)g}{M_1 + M_2 + \frac{M_p}{2}}$$

Pulley increases inertial mass, but "effective mass" is only ½ its total mass.

Physical Pendulum

Simple pendulum : mass has negligible size,
$m_{string} = 0$.

Let's review this before we consider the physical pendulum, for which these assumptions are not true.



$\cancel{ml\ddot{\phi} = -W \sin}$

Let's consider from the point of view of pure rotation about point of suspension:

$$I = ml^2$$

$$\tau = \cancel{W}\ \vec{r} \times \vec{F} \ \cancel{= -Wl \sin\phi}$$

$$\cancel{= lF} = -Wl \sin\phi$$

(since tends to make ϕ smaller).

$$\Rightarrow \quad \tau_i = I\alpha = I\ddot{\phi}$$

$$= -Wl\sin\phi$$

$$\Rightarrow \quad ml^2\ddot{\phi} = -mgl\sin\phi$$

$$\Rightarrow l\ddot{\phi} + g\sin\phi = 0$$

For small angle oscillations, $\phi \ll 1$,
$\sin\phi \simeq \phi$.

$\Rightarrow$

$$\Rightarrow \boxed{l\ddot{\phi} + g\phi = 0}$$

This is equation for simple harmonic motion

$$\phi = A \sin \omega t + B \cos \omega t \qquad \omega = \sqrt{g/L}.$$

$$t = 0, \quad \phi = \phi_0$$

$$\Rightarrow \phi = \phi_0 \cos \omega t.$$

Motion is <u>periodic</u>

Period given by $\omega T = 2\pi$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/L}} = 2\pi\sqrt{L/g}$$

Note: it is <u>independent</u> of amplitude ϕ_0!

This is great for a clock!

However, this is a consequence of $\sin\phi \simeq \phi$.

For finite ϕ_0, the period lengthens with amplitude.

<u>Physical Pendulum:</u>

Consider swinging object w/ mass M.

center of mass is distance l about the pivot.

Then it's <u>exactly</u> the same:

$$\tau = I\alpha \qquad (\text{before, } I_a = mL^2)$$

$$-lW\sin\phi = I_a\ddot{\phi}$$

$$\sin\phi \simeq \phi$$

$$\Rightarrow I_a\ddot{\phi} + Mlg\phi = 0.$$

Again, get eq$^{\underline{n}}$ of simple harmonic motion w/

$$\phi = A\cos\omega t + B\sin\omega t$$

$$\omega = \sqrt{\frac{MLg}{I_a}}$$

Better notation: let's <u>define</u> a radius of gyration k s.t.

$$I_o = Mk^2$$

moment of inertia about center of mass.

Hoop $k = R$
disk $k = \sqrt{\tfrac{1}{2}}\,R$
sphere $k = \sqrt{2/5}\,R$

By parallel axis thm:

$$I_a = I_o + ML^2$$
$$= M(k^2 + L^2).$$

$$\Rightarrow \omega = \sqrt{\frac{MLg}{I_a}} = \sqrt{\frac{gL}{k^2+L^2}}.$$

For simple pendulum, $k=0$

$$\Longrightarrow \omega = \sqrt{\frac{g}{L}}, \text{ as before.}$$