Week 3 - Pendulums + Rotational & Translational Motion

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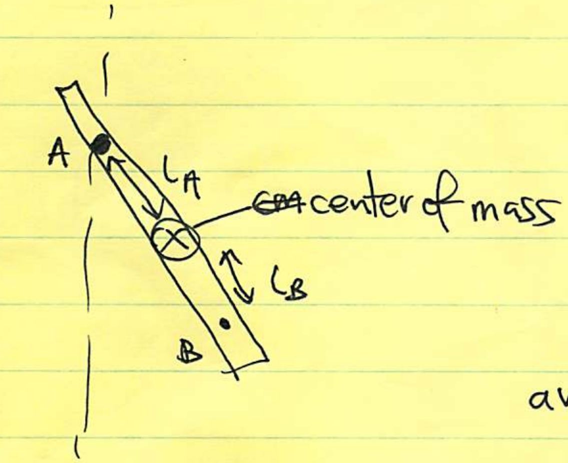
Kater's Pendulum



~~Previous~~: 16th – 20th Century most accurate measurement of g with pendulums.

$$T = 2\pi \sqrt{\frac{k^2 + L^2}{gL}}$$

average over many periods

limited by accuracy to which k (finite moment of inertia of weight) & L (due to uncertainty in c.m. ~~of~~ pendulum).

Kater's pendulum eliminates this uncertainty!

~~We just need to~~ Pendulum can be suspended fr. 2 adjustable edges, with period.

$$T_A = 2\pi \left(\frac{k^2 + l_A^2}{g l_A} \right)^{1/2}$$

$$T_B = 2\pi \left(\frac{k^2 + l_B^2}{g l_B} \right)^{1/2}$$

Adjust l_A, l_B until $T_A = T_B = T$.

Then:

$$\frac{k^2 + l_A^2}{g\, l_A} = \frac{k^2 + l_B^2}{g\, l_B}$$

$$\Rightarrow l_B\left(k^2 + l_A^2\right) = l_A\left(k^2 + l_B^2\right)$$

$$\cancel{\Rightarrow k^2 = \frac{l_B^2 - l_A^2}{l_B - l_A} = \frac{(l_B - l_A)(l_B + l_A)}{(l_B - l_A)}}$$

$$\Rightarrow k^2\left(l_B - l_A\right) = l_B l_A\,(l_B - l_A) \quad \cancel{= l_B + l_A}$$

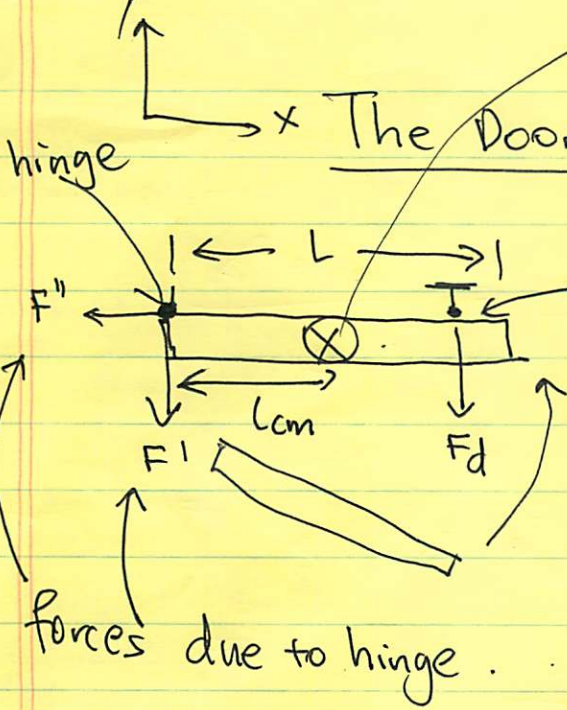
$$\Rightarrow k^2 = l_B l_A$$

$$\Rightarrow T = 2\pi\left(\frac{k^2 + l_A^2}{g\, l_A}\right)^{1/2}$$

$$= 2\pi\left(\frac{l_A l_B + l_A^2}{g\, l_A}\right)^{1/2} = 2\pi\left(\frac{l_B + l_A}{g}\right)^{1/2}$$

Only geometrical quantity is $l_A + l_B$,
which can be measured with great precision!
Don't need to know c.m...

The Doorstop



Banging of door against stop can tear off hinge. How to minimize this?

F'' = (small radial force)
→ gives door its centripetal acceleration.

~~F''~~ F', F_d = <u>large</u> forces which bring door to rest.

Forces on hinge equal & opposite to F', F''
↑ try to minimize this!

~~Const~~ To get expression for F', consider
- angular momentum of door about hinge
- linear momentum of c.m.

$$\tau = \frac{dL}{dt} \Rightarrow dL = \tau \, dt$$

$$\Rightarrow \quad \Delta L = \int_{t_i}^{t_f} \tau \, dt$$

~~&~~ ~~L_i~~ ~~$I\omega_0 = L$~~

$$I\omega_0 = l \int F_d \, dt \quad (1)$$

← integral is over duration of collision.

Similarly $F = \frac{dp}{dt}$

$$\Rightarrow \quad \Delta p = \int F \, dt$$

$$p_f = 0$$

$$p_i = m v_y = m l_{cm} \omega_0$$

$$F_y = -(F' + F_d).$$

$$p_f - p_i = \cancel{-} m l_{cm} \omega_0 = \cancel{-} \int (F' + F_d) \, dt$$

But from (1),

$$\int F_d \, dt = \frac{I \omega_0}{\cancel{4} L}$$

$$\Rightarrow \quad M l_{cm} \omega_0 - \frac{I \omega_0}{l} = \int F' \, dt$$

$$\Rightarrow \int F' \, dt = \omega_0 \left(M l_{cm} - \frac{I}{l} \right)$$

<u>Choose</u> the doorstop to be at a distance

$$l = \frac{I}{M l_{cm}}$$ → can <u>make impact force zero!</u>

For uniform ~~door~~ of width w,

$$I = \frac{M w^2}{3}$$ (just like for stick)

7i

$$\ell_{cm} = \frac{w}{2}.$$

$$\Rightarrow \ell = \frac{I}{M\ell_{cm}} = \frac{\frac{Mw^2}{3}}{M\frac{w}{2}}$$

$$= \frac{2}{3} w$$

Door stop must be at height of c-m. rather than at floor level → otherwise force on 2 hinges are unequal.

Dist. ℓ is center of percussion.

Just like hitting a sweet spot on a baseball bat
~~avoid~~ minimize impact

~~Mot~~ Combined Rotational & Translational Motion

Now relax assumption of pure rotation — consider rolling drum.

What is the appropriate axis to use?

Recall Chasle's theorem: any motion can be decomposed into translation of c.m. & rotation about c.m.

↳ So we will use c.m. coordinates!

fixed axis rotation:

Only consider motion where axis of rotation doesn't change direction in c.m. frame → call it the z axis.

Show that

rotation of body about center of mass

$$L_z = I_0\omega + (R \times MV)_z$$

motion of c.m. wrt origin of ~~co~~ inertial coordinate system

"spin angular momentum"

"orbital angular momentum"

~~E.g. think of earth's spin + orbital~~

E.g. think of earth's spin + orbital motion.

Will also encounter this in atomic physics → spin & orbital $\vec{L}$ of electron.

Proof:

$$\vec L = \cancel{\sum_{j=1}^{N} (\vec r_j \times m_j \overset{P_j}{\dot{\vec r}_j})} = \sum_j (\vec r_j \times m \overset{\vec p}{\dot{\vec r}_j})$$

(all wrt inertial coordinate system)

center of mass is described by a position vector:

$$\vec R = \frac{\sum m_j \vec r_j}{M}$$

Now use c.m. coordinates r_j':

$$\vec r_j = \vec R + \vec r_j{}'$$

Then

$$L = \sum_j (\vec r_j \times m_j \dot{\vec r}_j)$$

$$= \sum (\vec R + \vec r_j{}') \times m_j (\dot R + \dot{\vec r}_j{}')$$

$$= \overset{(1)}{\vec R \times \sum m_j \dot{\vec R}} + \overset{(2)}{\vec R \times \sum m_j \dot{\vec r}_j}$$

$$\cancel{+ \sum \vec r_j \times} \quad (3) \quad (4)$$

$$+ \sum m_j \vec r_j{}' \times \dot{\vec R} + \sum m_j \vec r_j{}' \times \dot{\vec r}_j{}'$$

Consider (1)

$$\vec R \times \sum m_j \dot{\vec R} = \vec R \times M\dot R = \vec R \times M\vec V$$

← velocity of c.m. wrt inertial system.

~~② $\vec{R} \times \sum m_j \dot{\vec{r}}_j = \vec{R} \times$~~

For terms ② & ④, consider

$$\sum m_j r_j' = \sum m_j (\vec{r}_j - \vec{R})$$

$$= M\vec{R} - M\vec{R}$$

$$= 0 .$$

Also $\sum m_j \dot{r}_j' = 0$ (since term is always zero, ~~#~~ it's a const and time derivative is zero).

So

② & ④ $= 0$.

Now we have:

$$\vec{L} = \vec{R} \times M\vec{V} + \sum \vec{r}_j' \times m_j \dot{\vec{r}}_j'$$

angular momentum due to c.m. motion

angular momentum due to motion about c.m.

~~Since~~ We are only considering fixed axis ROTATION! rotation (will consider arbitrary axis of rotation – e.g., gyroscopes, later).

Take z compont

$$L_z = (\vec{R} \times M\vec{V})_z + \left(\sum \vec{r}_j' \times m_j \dot{\vec{r}}_j'\right)_z$$

These terms are all evaluated in the c.m. frame.

So it's identical to the case of pure rotation we've already analysed!

$$\left(\sum m_j r_j' \times \dot{r}_j'\right)_z = \sum \left(m_j \vec{\rho}_j \times \dot{\vec{\rho}}_j'\right)$$

$\uparrow$ $\dot{\rho}_j' = \rho_j' \omega$

$$= \sum m_j \rho_j^2 \omega = I_0 \omega$$

$$L_z = I_0 \omega + (R \times M\vec{V})_z$$

$\uparrow$ angular momentum about c.m.

$\uparrow$ angular momentum about origin

$\boxed{\text{QED!}}$

Two things to note:

→ this expression still valid if the c.m. is accelerating, since $\vec{L}$ is calculated wrt inertial coord. system.

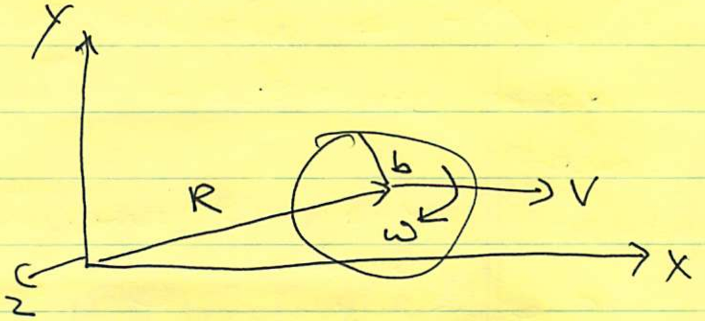
→ spin angular momentum is independent of coord system (no change ~~in~~ of coord system can eliminate spin) but orbital angular momentum disappears if ~~the~~ origin is c.m.

Week 3 Rotation + Translation (Con't) ~~Angular~~ ①

Angular Momentum of a Rolling Wheel.

Calculate angular momentum of a rolling wheel of mass M, radius b which rolls uniformly & w/out slipping.

about center of mass → $I_0 = \frac{1}{2} M b^2$



$L_0 = -I_0\omega\,\hat{z}$
$= -\frac{1}{2} M b^2 \omega\,\hat{z}$

$(R\times MV)_z$ ~~$= -MbV$~~ ~~(since $R_\perp$~~
~~$= R_y$~~
[illegible]
$= -MR_yV_{\cancel{x}}$

$= -MbV$

$L_z = I_0\omega + (\vec{R}\times M\vec{V})_z$
$= -\frac{1}{2}Mb^2\omega - \underbrace{MbV}_{Mb(b\omega)}$

$V = b\omega$ if no slip

~~≠ ⇒~~

$= -\frac{3}{2}Mb^2\omega$

Tor~~que~~:

$$\cancel{\tau = \sum r_j \times f_j}$$

Torque also divides into 2 components:

$$\tau = \sum \vec{r_j} \times \vec{f_j}$$

$$= \sum (\vec{r_j}' + \vec{R}) \times \vec{f_j}$$

$$= \sum (r_j' \times f_j) + \cancel{\vec{R} \times} \underbrace{\sum R \times f_j}_{\vec{R} \times \vec{F} \; (= \sum f_j)}$$

~~torqu~~ torque about center of mass

torque due to total external force acting at c.m.

For ~~fixed~~ axis rotation, $\omega = \omega \hat{k}$

$$\Rightarrow \tau = \tau_0 + (R \times F)_z$$

But from ~~dt~~ $L_z = I_0 \omega + (\vec{R} \times M\vec{V})_z$,

$$\frac{dL_z}{dt} = I_0 \frac{d\omega}{dt} + \frac{d}{dt}(\vec{R} \times M\vec{V_z})$$

$$= I_0 \alpha + (\vec{R} \times M\vec{a})_z$$

$$\Rightarrow \tau_0 + (\vec{R} \times \vec{F})_z = I_0 \alpha + (R \times M\vec{a})_z$$

$$= I_0 \alpha + (R \times \vec{F})_z$$

(using $\vec{F} = M\vec{a}$)

~~⇒~~

$$\Rightarrow \boxed{\tau_0 = I_0 \alpha.}$$

i.e. $\tau_0 = I\alpha$ is true even if c.m. is accelerating; rotational motion about c.m. depends only on torque about c.m., independent of translational motion.

~~Final der~~ Finally, let's look at kinetic energy:

$$K = \frac{1}{2}\sum m_j v_j^2$$

motion about c.m.

motion of c.m.

$$= \frac{1}{2}\sum m_j(\dot{\rho}'_j + V)^2$$

$$= \frac{1}{2}\underbrace{\sum m_j \rho_j'^2}_{I}\,\omega^2 + \underbrace{\sum m_j \dot{\rho}'_j \cdot \vec{V}}_{0} + \frac{1}{2}\sum m_j V^2$$

since $= r'_{cm} = 0$

$$= \frac{1}{2} I_0 \omega^2 + \frac{1}{2} MV^2$$

↑ kinetic energy of spin

↑ orbital center of mass motion

Summary: Formulae for Fixed Axis Motion

Pure rotation: no translation

$$L = I\omega$$
$$\tau = I\alpha$$
$$K = \frac{1}{2} I\omega^2.$$

Rotation & translation (subscript 0 refers to c.m.)

$$L_z = I_0\omega + (R \times MV)_z$$

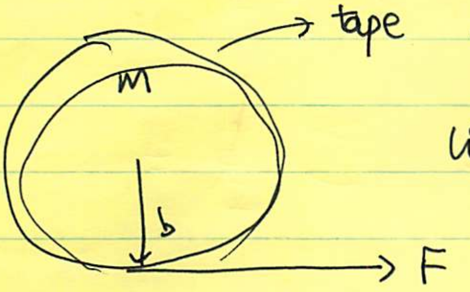
$$\tau_z = \tau_0 + (\vec{R} \times \vec{F})_z$$

$$\tau_0 = I_0 \alpha$$

$$K = \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M V^2 .$$

We are now just going to work a bunch of examples

Disk on Ice Pull by const force by



tape wound about circumference.
Disk slides on ice w/out friction.
What is its motion?
~~(this depends on the origin (that we choose)~~.

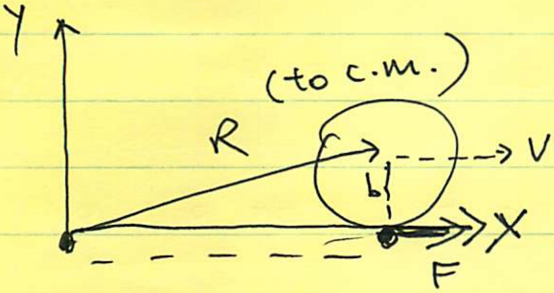
We can analyse the torque & angular momentum using different origins. ~~that~~
→ they will be origin dependent.
→ but a, α are not ~~or~~ frame dependent.

~~Acce~~ $\boxed{a = \frac{F}{M}.}$ (in all frames).

Analyze in frame of center of mass.

$$\tau_0 = bF$$
$$= I_0 \alpha$$
$$\Rightarrow \boxed{\alpha = \frac{bF}{I_0}.}$$

Let's check that we get the same answer in another frame



$$\tau_z = \tau_0 + (\vec{R}\times\vec{F})_z$$
$$= bF \underset{-bF}{\cancel{\times bF}} = 0$$

As expected, since $\vec{r}$ is ‖lel to $\vec{F}$.

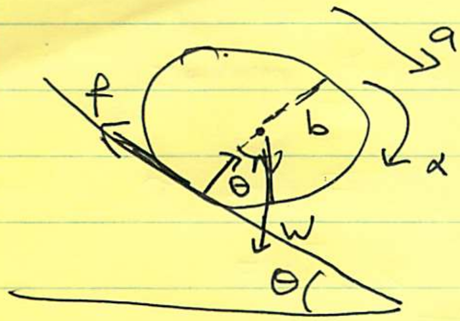
Hence, $\vec{L}$ is conserved.

$$L_z = I_0\omega + (\vec{R}\times M\vec{V})_z$$
$$= I_0\omega - bMV$$

$$\frac{dL_z}{dt} = I_0\alpha - bMa = 0$$

$$\Rightarrow \quad \alpha = \frac{bMa}{I_0} = \frac{bF}{I_0}$$ (as before).

Drum Rolling down a plane



Uniform drum, rolls w/out slipping down an inclined plane

$$I_0 = \frac{Mb^2}{2}$$

Want to solve for acceleration along plane.

Again, solve 2 ways, using different coord system.

~~Trans~~ Motion of c.m. along plane:

$$W\sin\Theta - f = Ma. \qquad (1)$$

Rotation about c.m.

$$bf = I_0\alpha \quad \Rightarrow f = \frac{I_0\alpha}{b}. \qquad (2)$$

For rolling w/out slipping,

$$a = b\alpha. \qquad (3)$$

~~Eliminate~~ Plug (2) into (1),

$$W\sin\Theta - \frac{I_0\alpha}{b} = Ma$$

$$\Rightarrow W\sin\Theta - \frac{\cancel{I_0\alpha}}{b}\left(\frac{a}{b}\right)\left(\frac{Mb^2}{2}\right) = Ma$$

$$\Rightarrow \cancel{M}g\sin\Theta - \frac{\cancel{M}a}{2} = \cancel{M}a$$

$$\Rightarrow \boxed{a = \frac{2}{3}g\sin\Theta}$$

(N.B. if no rotation, would be ~~g~~ $a = g\sin\Theta$).

∃ more complicated way of doing it, DEFER!