Week 4

HRK 12-6

①

Combined Rotational & Translational Motion

Yo-Yo

Let's figure out how fast a yoyo has to spin ~~to reach~~ at the bottom to reach our hand.

From conservation of energy,

$$\frac{1}{2} I \omega_0^2 = \frac{1}{2} I \omega^2 + \frac{1}{2} M v^2 + MgL$$

$\uparrow$ spin at bottom

$\uparrow$ at top, have combination of rotational, translational k.e. + potential energy.

Minimum rotation is when have only ~~rotational k.e~~ potential energy at top:

$$\Rightarrow \cancel{\frac{1}{2}\left(\frac{2}{5}MR^2\right)\omega_0^2 = MgL}$$

$$\Rightarrow \cancel{\omega_0 =} \frac{1}{2}\left(\frac{1}{2}\cancel{M}R^2\right)\omega_0^2 = \cancel{M}gL$$

$$\Rightarrow \omega_0 = \sqrt{\frac{4gL}{R^2}} = \frac{2}{R}\sqrt{gL}$$

$L \sim 1\text{m}$.
$R \sim 3\text{cm}$.

$$\omega_0 \sim \frac{2}{0.03}(10\times1)^{1/2}$$

$$\sim \frac{2}{0.03}\,3 \;\sim 200 \text{ rad s}^{-1}$$

$$\sim \frac{200}{2\pi} \sim 33 \text{ rev s}^{-1}$$

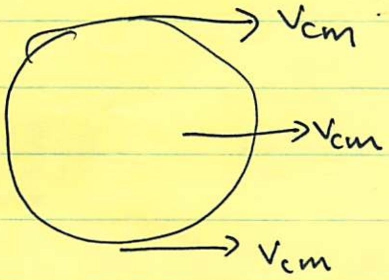
~~ta~~ This is lower limit.
> 100 rev s^{-1} is possible, esp if thrown downward w/ force.

~~Consider rolling wit~~

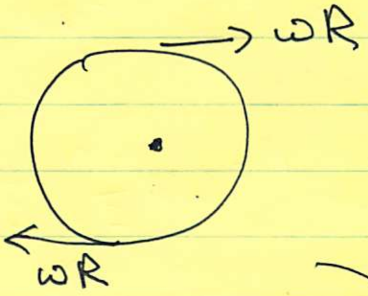
ROLLING WITHOUT SLIPPING

Show photo of bicycle wheel → sharp on bottom, blurry on top.

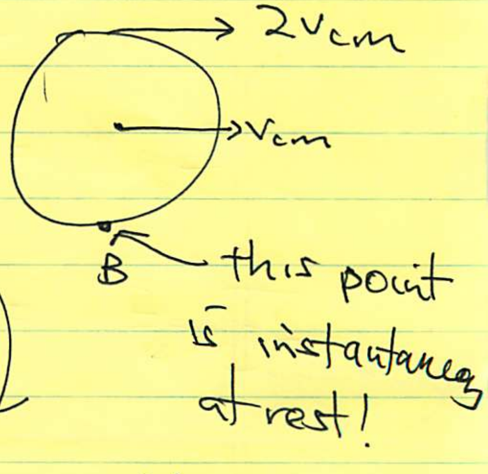
~~Con~~ Consider case of rolling without slipping – can be decomposed into translational & rotational motion.



translation



rotational



condition of rolling w/out slipping → ~~point~~ no relative motion between wheel & ground → frictional force does no work (contrast w/ case where wheels slip on snow → frictional force melts ice)

~~$\Rightarrow v_{cm} = \omega R$~~.

$$\Rightarrow v_{cm} - \omega R = 0 \Rightarrow v_{cm} = \omega R.$$

ONLY true for rolling w/out slipping.

~~for rolling w/out slip~~

In general, ~~k.e. &~~ rotational & translational energies are independent:

$$K = \frac{1}{2} I \omega^2 + \frac{1}{2} M v_{cm}^2$$

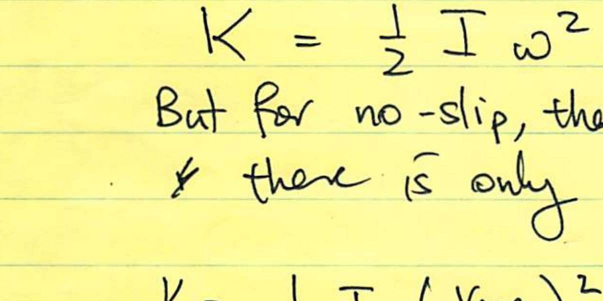
But for no-slip, they are related and ~~&~~ there is only 1 independent term:

$$K = \frac{1}{2} I \left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} M v_{cm}^2$$

OR

$$K = \frac{1}{2} I \omega^2 + \frac{1}{2} M (\omega R)^2.$$

Another way of viewing rolling w/out slipping:



Consider point of contact to be instantaneous axis of rotation → at each point in time, ~~At each instant there~~ have pure rotational ~~ro~~ motion!

~~At each~~, At each instant there is a new axis of rotation.

moment of inertia about B

$$K = \frac{1}{2} I_B \omega_B^2$$

angular velocity about B.

From parallel axis theorem,

$$I_B = I_{cm} + MR^2$$

$$= \frac{3}{2} MR^2 .$$

$\omega_B R = v_{cm}$ (since must give right velocity for c.m.)

$$\Rightarrow K = \frac{1}{2}\left(\frac{3}{2} MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \frac{3}{4} M v_{cm}^2$$

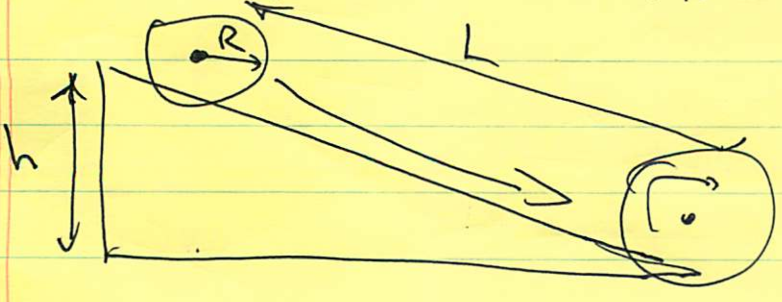
→ same ans as before!

~~$= \frac{3}{4}$~~

$$\Big[= \frac{1}{2} M v_{cm}^2 + \frac{1}{2}\left(\frac{1}{2} MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 \Big)$$

$$= \frac{3}{4} M v_{cm}^2 .$$

This derivation has been done by assuming rotational & translational motion aren't independent



~~Cyl~~ Cylinder of mass M, radius R. rolling down inclined plane. What is speed at bottom?

Use conservation of energy:

$$\frac{1}{2} M v_{cm}^2 + \frac{1}{2}\left(\frac{1}{2} M R^2\right)\left(\frac{v_{cm}}{R}\right)^2 = Mgh$$

$$\Rightarrow \quad \frac{3}{4} \cancel{M} v_{cm}^2 = \cancel{M} g h$$

$$\Rightarrow \quad v_{cm} = \sqrt{\tfrac{4}{3} g h}$$

If ~~no~~ sliding w/out friction, then.

$$\frac{1}{2} \cancel{M} v_{cm}^2 = Mgh$$

$$\Rightarrow \quad v_{cm} = \sqrt{2gh} \qquad \underline{\text{faster}}$$

They have the same <u>total</u> amt. of kinetic energy (= mgh),
just in one case, all translational (~~no~~ sliding)
in another, translational + rotational (~~slide~~ rolling).

Solve the same problem w/ ~~energy~~ dynamical methods based on forces & torques.

→ more complicated & less direct

→ but, if you want to know values of forces (e.g. frictional force), have to use dynamical method.

<u>Note:</u> usually a problem can be solved

~~A~~ many different ways.

Use

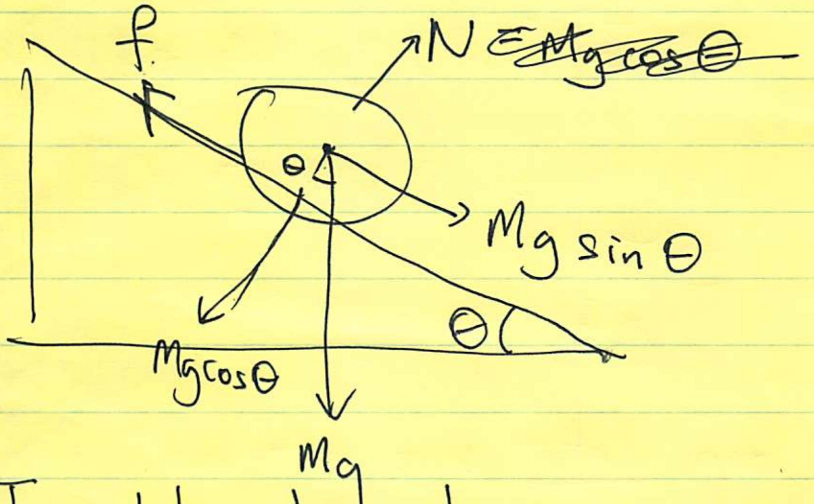
$$\tau_{total} = \Sigma \tau = I \alpha$$

about axis through c.m.

Recall: this is true even if c.m. is accelerating.

True if ① axis about c.m.

② axis doesn't change direction in space.



Translational motion

Motion ⊥ to incline

(1) $$N - Mg \cos\theta = 0$$ (since no motion ⊥ to inclinde).

Motion ∥ to incline

(2) $$Mg \sin\theta - f = Ma_{cm}$$

Rotational motion

3 forces acting on cylinder: N, Mg, f.

N & Mg pass through c.m. → zero moment arm → NO TORQUE.

$$\tau = fR = I_{cm}\alpha$$
$$= I_{cm}\left(\frac{a}{R}\right)$$

$$\underset{(3)}{\Rightarrow} f = I_{cm}\left(\frac{a}{R^2}\right) = \frac{1}{2}MR^2\left(\frac{a}{R^2}\right)$$
$$= \frac{1}{2}Ma_{cm}.$$

Plug (3) into (1),

$$\cancel{M}g\sin\theta - \frac{1}{2}\cancel{M}a_{cm} = \cancel{M}a_{cm}$$

$$\Rightarrow a_{cm} = \frac{2}{3}g\sin\theta.$$ (vs. $a_{cm} = g\sin\theta$ if ~~not~~ ~~no~~ sliding, not rolling)

This is constant, independent of cylinder's position along incline.

~~$\dot{v} = a$~~

~~$\Rightarrow \frac{dv}{dt} = \Rightarrow v\,v = av$~~

For const acceleration,

$$v = v_0 + at$$ ~~$\Rightarrow v = at$~~

$$x = x_0 + v_0 t + \frac{1}{2}at_0^2$$

But $v_0 = 0$, set $x_0 = 0$.

$\Rightarrow$

$$v = at$$
$$x = \frac{1}{2}at^2 \Rightarrow x = \frac{1}{2}a\left(\frac{v}{a}\right)^2$$
$$= \frac{1}{2}v^3/a$$

$$\Rightarrow \quad v^2 = 2a\cancel{x}$$

$$\Rightarrow \quad v_{bot}^2 = 2aL$$

$\sin\theta = \frac{h}{L}$

$$= 2\left(\tfrac{2}{3} g \sin\theta\right) L$$

$$= \tfrac{4}{3} g \left(\tfrac{h}{L}\right) L$$

$$\Rightarrow \quad v_{bot} = \sqrt{\tfrac{4}{3} g h} \quad \checkmark$$

same as before.

But now, we get ~~the coefficient of static friction:~~ force of static friction need for rolling

$$f = \frac{M a_{cm}}{2} = \frac{M}{2}\left(\tfrac{2}{3} g \sin\theta\right)$$

$$= \tfrac{1}{3} M g \sin\theta$$

$$\Rightarrow \quad f = \mu_s N$$

$$\Rightarrow \quad \mu_s N > \tfrac{1}{3} M g \sin\theta$$

$$\Rightarrow \quad \mu_s \cancel{Mg} \cos\theta > \tfrac{1}{3} \cancel{Mg} \sin\theta$$

$$\Longrightarrow \quad \mu_s > \tfrac{1}{3} \tan\theta$$

if smaller than this, will start to slip!

Sphere, cylinder & hoop start fr. rest & roll down incline. Which gets to the bottom first?

From before,

$$Mg\sin\theta - f = Ma_{cm}$$

$$f = \frac{I_{cm}\alpha}{R} = I_{cm}\frac{a_{cm}}{R^2}$$

$$\Rightarrow Mg\sin\theta - \frac{I_{cm}a_{cm}}{R^2} = Ma_{cm}$$

$$\Rightarrow a_{cm} = \frac{Mg\sin\theta}{\left(M + \frac{I_{cm}}{R^2}\right)}$$

For sphere $I = \frac{2}{5}MR^2 \Rightarrow a_{cm} = \frac{g\sin\theta}{(1+2/5)}$

$$= 5/7\, g\sin\theta$$

For cylinder $I = \frac{1}{2}MR^2 \Rightarrow a_{cm} = \frac{g\sin\theta}{(1+\frac{1}{2})}$

$$= 2/3\, g\sin\theta$$

For Hoop $I = MR^2 \Rightarrow a_{cm} = \frac{1}{2}g\sin\theta$.

$\longrightarrow$ in order.

sphere, cylinder, hoop

Ask: which has the most KE?
They <u>all</u> have the same KE, just differently divided up between rotational & translational.
Spher is most "compact", most c.m. motion.

Can also solve by energy methods:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

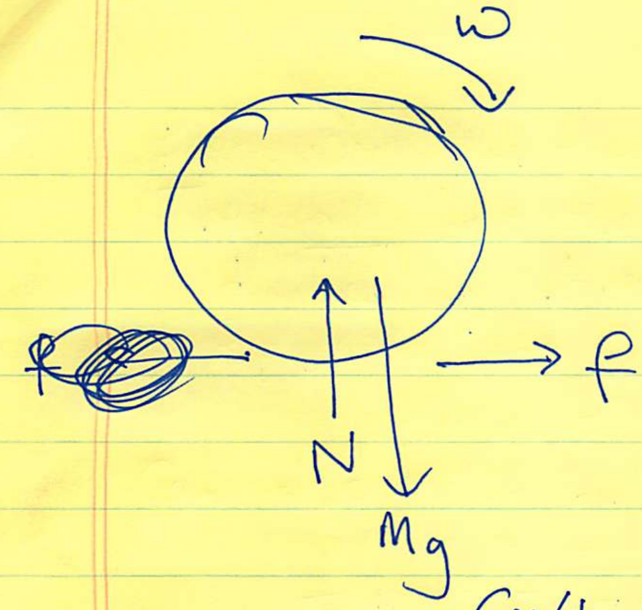
$$\rightarrow \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = mgh$$

~~⇒ so~~

$$\Rightarrow v = \frac{2gh}{\left(m + \frac{I}{R^2}\right)}$$ → solve for v.

Suppose I have a marble & a bowling bowl, and align it so that their c.m. is in the same position. Which will reach the bottom first?
→ they'll reach it at the same time! Note that a_{cm} does <u>not</u> depend on M or R.

<u>Problem</u> Cylinder of mass M, radius R and spins with angular velocity ω_0. Lowered onto surface w/ coefficient of friction μ_k. After a time t it rolls w/out slipping.
What is v_{cm} when it rolls w/out slipping, & what is time t?



$$f = \mu_k N = M a_{cm}$$

$$\cancel{f} \Rightarrow \mu_k M g = M a_{cm}$$

$$\rightarrow a_{cm} = \mu_k g$$

~~Can't do conservation of~~

~~Can't use~~

~~While finally rolls w/out slipping, $v_f = \omega R$~~

~~a_{cm}~~ ⇒ So final velocity is

$$v_f = a_{cm} t = \mu_k g t$$

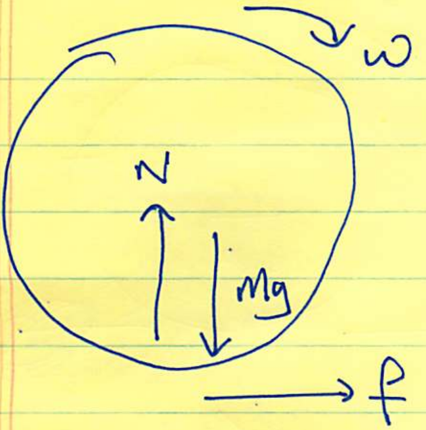
~~whe~~

Angular acceleration

$$I\alpha = fR \Rightarrow \cancel{I\left(\frac{v_f}{Rt}\right) = \mu_k g R}$$

$$\Rightarrow \cancel{v_f = \frac{\mu_k g R^2 t}{I}}$$

$$\Rightarrow \cancel{t = 2v_f}$$



$$f = \mu_k N = M a_{cm}$$

$$\mu_k M g = M a_{cm}$$

N.B.

$$\Rightarrow a_{cm} = \mu_k g$$ <u>constant</u>

$$\Rightarrow \frac{v_f}{t} = \mu_k g$$

$$\Rightarrow \frac{1}{t} = \frac{\mu_k g}{v_f}$$

Angular acceleration

$$\tau = I \alpha$$

$$f R = I \left[\frac{\omega_f - \omega_i}{\cancel{t}} \right]$$

$$\Rightarrow \frac{1}{t} \left[-\frac{v_f}{R} + \omega_0 \right] I = \mu_k M g R$$

$$\Rightarrow \frac{\cancel{\mu_k g}}{v_f} \left[-\frac{v_f}{R} + \omega_0 \right] I = \cancel{\mu_k} M \cancel{g} R$$

$$\Rightarrow \left[-\frac{1}{R} + \frac{\omega_0}{v_f} \right] \frac{\cancel{M} R^{\cancel{2}}}{2} = \cancel{M} \cancel{R}$$

$$\Rightarrow -1 + \frac{\omega_0 R}{v_f} = 2$$

$$\Rightarrow \boxed{v_f = \frac{1}{3} \omega_0 R}$$

Does <u>not</u> depend on M, g or μ_k!
But nothing would happen if there were zero → no friction...

$$\Rightarrow \quad t = \frac{v_f}{\mu_k g} = \frac{1}{3}\frac{\omega_0 R}{\mu_k g}.$$

Work energy Theorem (section 6.7 KK)

Work energy theorem for particle

$$\cancel{\Delta W}\ \Delta K = W = \oint F.dr.$$

Show that work energy theorem can be written as

$$\Delta K = \Delta\left(\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2\right) = \oint F.dr + \oint \tau\, d\theta$$

For translational part:

$$F = Ma = m\frac{dv}{dt}$$

$$\cancel{\Rightarrow} \text{ If } dr = v\,dt$$

$$\Rightarrow F\,dr = \left(m\frac{dv}{dt}\right)v\,dt$$

$$= d\left(\frac{1}{2}mv^2\right)$$

Integrate

$$\oint_{R_a}^{R_b} F.dr = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

Consider work done by rotational k.e.

$$\tau_0 = I_0\alpha$$

$$= I_o \frac{d\omega}{dt}$$

$\cancel{\Leftrightarrow}$ if $d\theta = \omega dt$

$$\Rightarrow \tau_o d\theta = (\omega dt)(I_o)\frac{d\omega}{dt}$$

$$= d\left(\frac{1}{2} I_o \omega^2\right)$$

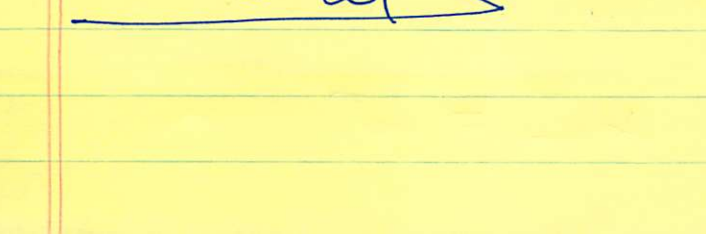
$$\Rightarrow \int_{\theta_a}^{\theta_b} \tau_o d\theta = \frac{1}{2} I_o \omega_b^2 - \frac{1}{2} I_o \omega_a^2$$

$$\Rightarrow K_b - K_a = W_{ba}$$

total ↗ k.e. ↑ total work done by forces & torques.

<u>Work-energy theorem is 2 independent theorems</u>, one for translation + 1 for rotation.

Drum rolling down Plane: Energy Method



Consider uniform drum rolling down a plane.