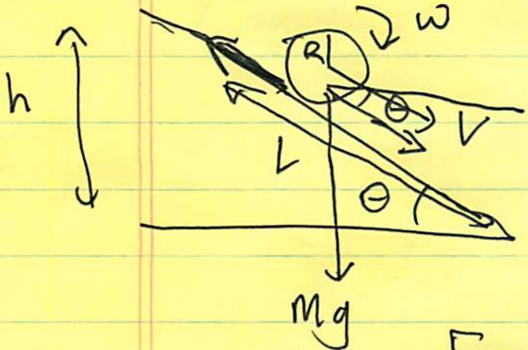
Week 5 - Rotational Motion Wrap-Up & Review

Solving Problems Using Energy Methods

(K&K 6.7)

Drum Rolling down plane



Using work-energy theorem to solve for speed of cylinder ~~falling~~ rolling without slipping down inclined plane.

~~For~~ Translational motion:

$$\int_a^b F\cdot dr = \frac{1}{2}M v_b^2 - \cancel{\frac{1}{2} M v_a^2} \rightarrow 0 \qquad v_a = 0$$

$$\Rightarrow \quad (Mg\sin\theta - f)\,L = \frac{1}{2}Mv^2. \qquad (1)$$

Rotational Motion:

$$\int_{\theta_a}^{\theta_b} \tau\, d\theta = \frac{1}{2} I_0 \omega_b^2 - \cancel{\frac{1}{2} I_0 \omega_a^2} \rightarrow 0 \qquad \omega_a = 0$$

$$\Rightarrow \quad fR\theta = \frac{1}{2} I \omega^2$$

For rolling w/out slipping $R\theta = L$.

$$\Rightarrow \quad fL = \frac{1}{2} I \omega^2 = \frac{1}{2} I \left(\frac{V}{R}\right)^2 \qquad (2)$$

Back to equation (1)

$$Mg\underbrace{L\sin\theta}_{h} - fL = \frac{1}{2}Mv^2$$

$$\Rightarrow \quad Mgh = \frac{1}{2} I \left(\frac{V}{R}\right)^2 + \frac{1}{2} M v^2$$

$$I = \frac{1}{2} M R^2$$

$$\cancel{= \frac{1}{2} M (}$$

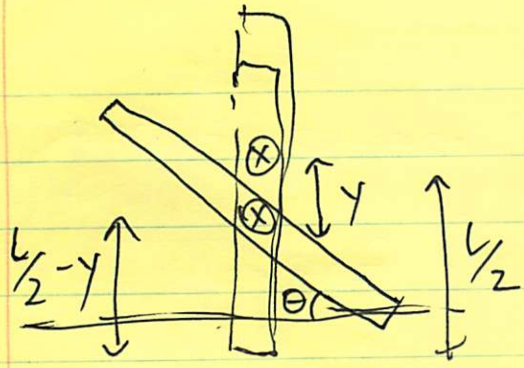
$$= \frac{1}{2} M \left[\frac{1}{2} v^2 + v^2\right]$$

$$= \frac{3}{4} M v^2$$

$$\Rightarrow \quad \boxed{V = \sqrt{\tfrac{4}{3} g h}}$$

Friction ~~does no work~~ is not dissipative in this case: from eq(1), decreases translational motion by amount fL, but increases rotational motion by same amt. → simply converts ~~1 form of~~ translational → rotational K.E.

~~Ex~~ If there is slipping, no longer true, friction heats surface.



Falling Stick Initially upright stick falls on frictionless table. Find speed of c.m. as function of position.

Since there are no horizontal forces, c.m. must fall straight down. → use energy methods!

Initial energy : $E = \frac{MgL}{2}$.

Later, it is $E = \underbrace{\frac{1}{2} I \dot{\theta}^2}_{\text{rotational}} + \underbrace{\frac{1}{2} M \dot{y}^2}_{\text{translational}} + Mg\left(\frac{L}{2} - y\right)$

Conservation of energy:

$$\frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} M \dot{y}^2 + Mg\left(\frac{L}{2} - y\right) = \frac{MgL}{2}$$

We have everything except $\dot{\theta}$ → how to get this?
Constraint equation

$$\left(L/2 - y\right) = L/2 \cos\theta$$

$$\Rightarrow -\dot{y} = -L/2 \sin\theta \, \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{2\dot{y}}{L\sin\theta}$$

For a stick, about the c.m., $I = \frac{ML^2}{12}$.

$$\frac{1}{2} M \dot{y}^2 + \frac{1}{2}\left[\frac{ML^2}{12}\right]\left[\frac{2\dot{y}}{L\sin\theta}\right]^2 + Mg\left(\frac{L}{2} - y\right) = Mg\frac{L}{2}$$

$$\Rightarrow \dot{y}^2\left[\cancel{\tfrac{1}{2}}\,1 + \frac{1}{3}\frac{1}{\sin^2\theta}\right] = 2gy$$

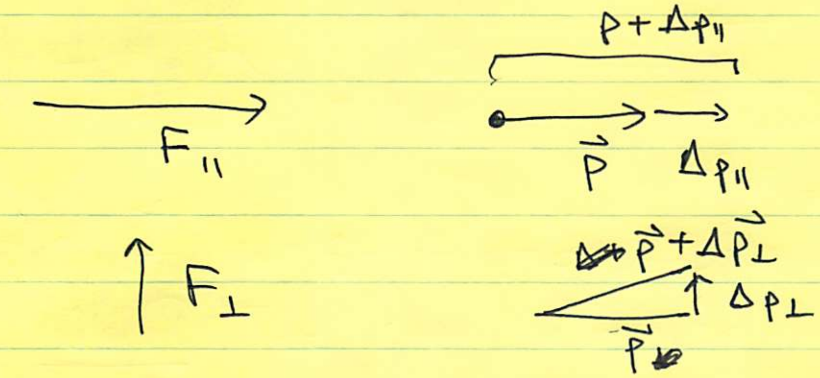
$$\Rightarrow \dot{y} = \left[\frac{2gy}{1\,\cancel{\tfrac{1}{2}} + \frac{1}{3}\frac{1}{\sin^2\theta}}\right]^{1/2}$$

$$\cancel{= \left[\frac{6gy}{3 + \sin^2}\right.}$$

$$= \left[\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}\right]^{1/2}.$$

~~To~~ IMPORTANT ASIDE (HRK 13-2)

Torque = changing magnitude & direction of $\vec{L}$



Suppose ~~a~~ force $\vec{F}$ acts on particle moving w/ momentum $\vec{p}$.

Can be resolved into $F_{\parallel}$, $F_{\perp}$

$F_{\parallel}$: changes magnitude but not direction

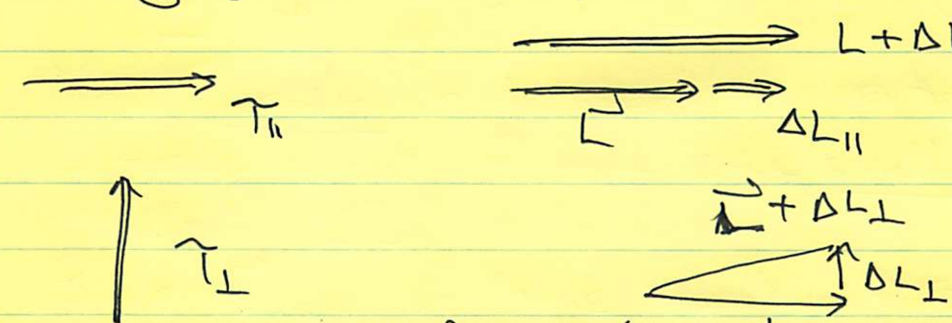
$F_{\perp}$: changes direction but not magnitude (if $|\Delta p_{\perp}|$ small)

we already know an example of this: centripetal force (F is always $\perp$s to tangential velocity).

Same thing goes for torques: ~~can~~ can resolve into $\parallel$el & $\perp$ component

This is harder to understand

c.f. conical pendulum



change in magnitude is small:

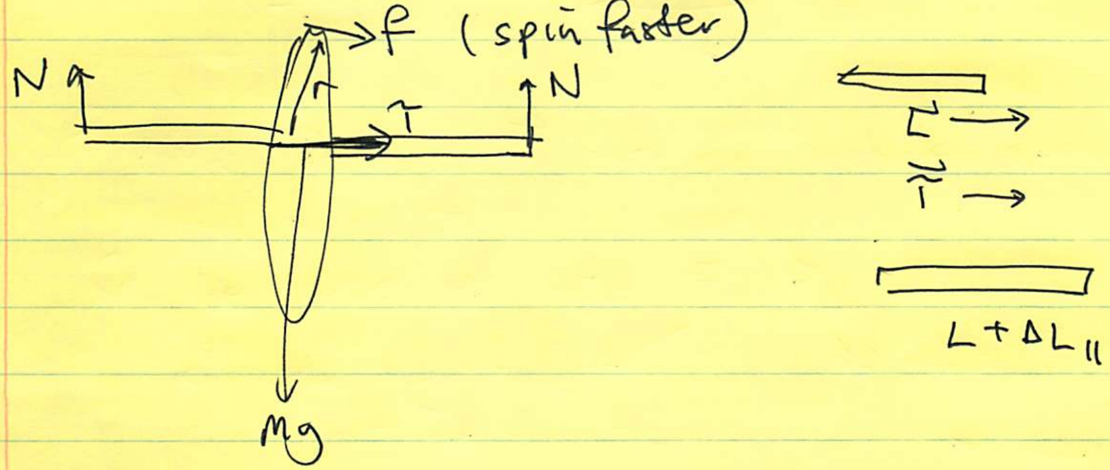
$$(\vec{L} + \Delta\vec{L}_{\perp})^2 = L^2 + \underbrace{2L \cdot \Delta L_{\perp}}_{\to 0 \text{ since } \perp} + \Delta L_{\perp}^2$$

$\Delta L_{\perp}^2$: 2nd order in small quantity.

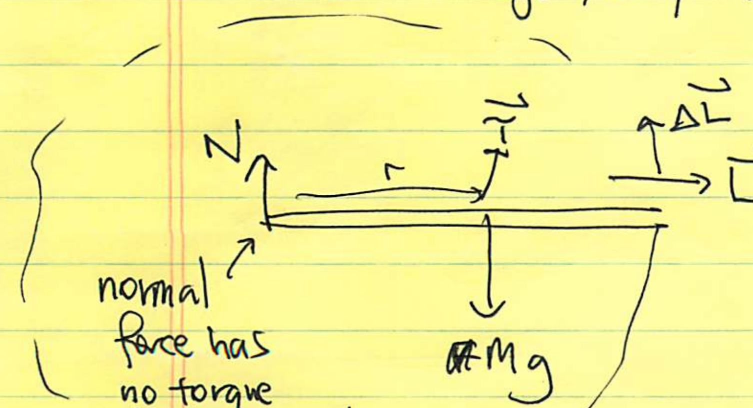
No work is done if $\vec{F} \perp \vec{p}$ or $\vec{\tau} \perp \vec{L}$

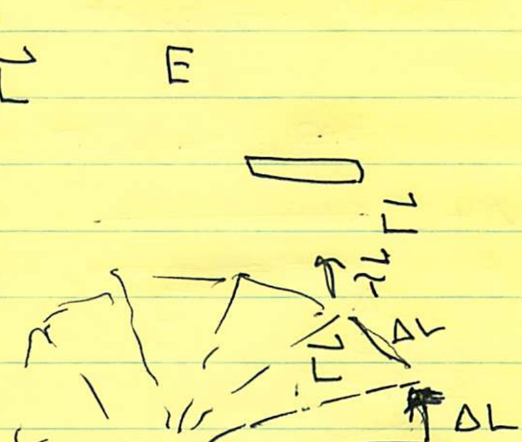
↳ hence no change in K.E., & motion continues at same linear or rotational speed

DEMONSTRATION First spin faster



Now let go, only force acting is gravity.



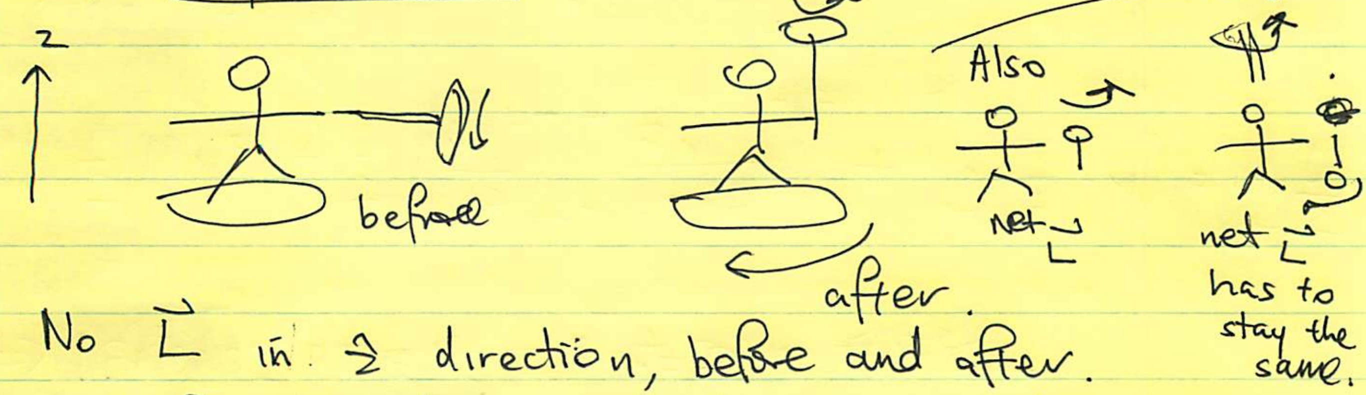


Gravity causes wheel to spin around on pivot!

ROTATING BICYCLE WHEEL

~~Gyroscope~~ [DEMONSTRATION]

this causes a bigger ~~change~~

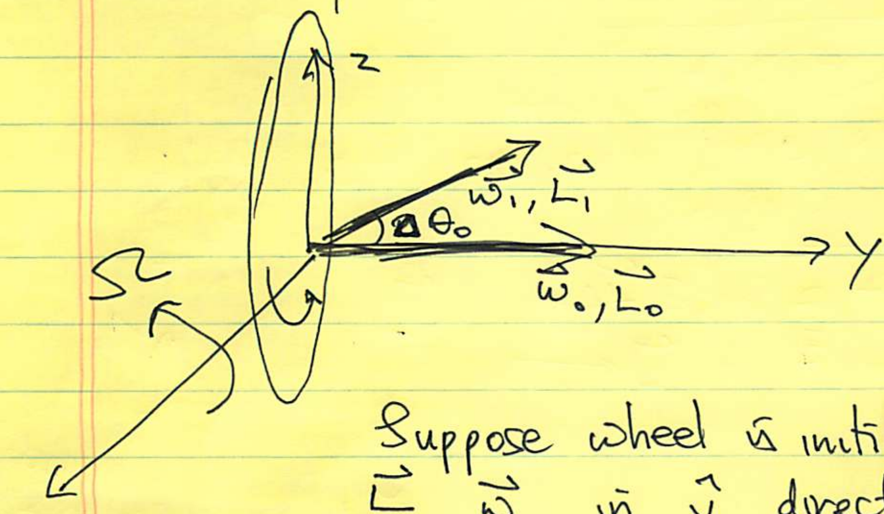


No $\vec{L}$ in $\hat{z}$ direction, before and after.

Why? The chair is frictionless, so there are no external torques on system.

So by conservation of $\vec{L}$, chair has to rotate once bring wheel to vertical.

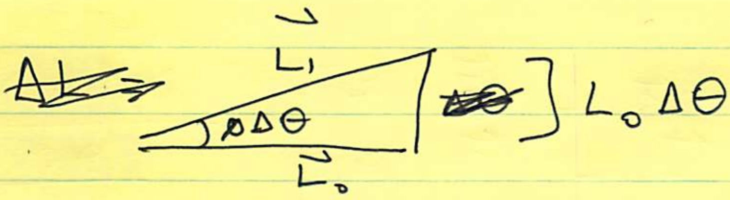
Let's try to understand this in more detail.



Suppose wheel is initially spinning about y-axis. $\vec{L}$, $\vec{\omega}$ in $\hat{y}$ direction.

Now slowly rotate wheel to vertical ($\hat{z}$ direction) by rotating about $\hat{x}$ axis with small angular velocity Ω.

Since $\vec{L}_{spin}$ is large, ~~angular~~ this only produces a small change: $\vec{L}$ changes in direction, not in magnitude.

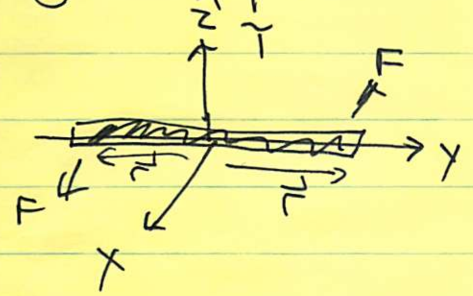


$$\Delta L = L_0 \Delta\Theta \Rightarrow \tau = \frac{\Delta L}{\Delta t} = L_0 \frac{\Delta\Theta}{\Delta t} = L_0 \Omega$$

To get order ~~...~~ right $\vec{\tau}$ is in $\hat{z}$ direction (since ΔL is in $\hat{z}$ direction)

$$\Rightarrow \boxed{\underset{(\hat{z})}{\vec{\tau}} = \underset{(\hat{x})}{\vec{\Omega}} \times \underset{(\hat{y})}{\vec{L}_0}}$$

To get torque in $\hat{z}$ direction, need



$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\hat{z} \qquad \hat{y} \times (-\hat{x})$$
$$- \qquad -\hat{y} \times (\hat{x})$$

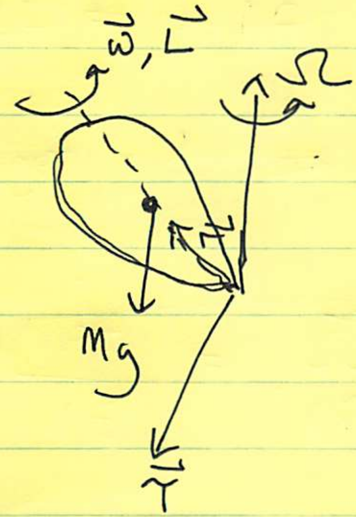
So to move bicycle wheel to <u>vertical</u> direction need to ~~move~~ apply force in <u>horizontal</u> direction ($-\hat{x}$, $\hat{x}$ direction)

These forces are applied by us → so by Newton's 3rd law, ~~these~~ there are equal & opposite forces/torques acting on us → we spin about!

We can generalize these results:

show spinning top.

SPINNING TOP [show bicycle top].



$\vec{\tau} = \vec{r} \times \vec{F}$ (out of page)

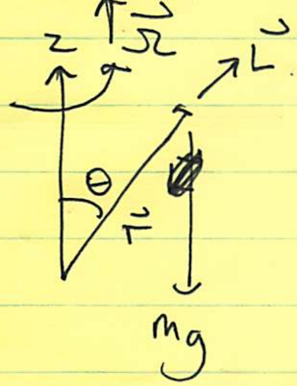
$\Rightarrow \Delta L$ is towards us

<u>precesses</u> in a circle around z axis

$\vec{\tau} = \vec{\Omega} \times \vec{L}_0$

angular velocity of precession

We can figure out the angular velocity of precession:



$$\vec{\tau} = \vec{\Omega} \times \vec{L}_0$$

$$|\tau| = \Omega L \sin\theta = \cancel{r \times F}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$= r M g \sin\theta$$

$$\Rightarrow r M g \sin\theta = \Omega L \sin\theta$$

$$\Rightarrow \Omega = \frac{rMg}{L}$$

the <u>faster</u> the top is spinning (larger $\vec{L}$), the <u>slower</u> the precession.

Also, as $r\downarrow$, $\Omega\downarrow$ (since gravity exerts less torque).

Play with bicycle wheel: spin faster or slower (change $\vec{L}$)
→ how does Ω change?
Play with Maxwell top: change $\vec{r}$ → how does Ω change?

~~These Note that $\Omega \propto Mg \propto \vec{F}$ (applied force)~~

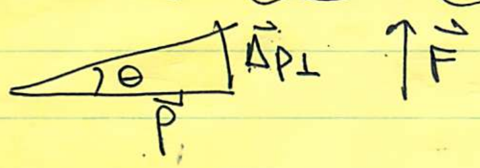
Note ~~that~~

since ~~$\Omega \propto \frac{rF}{L} \propto \frac{\tau}{L}$~~

~~$\Omega \propto$ $\vec{\tau} = \vec{\Omega} \times \vec{L}$~~

These considerations ⇒ have important consequences for directional stability.

Translational motion:



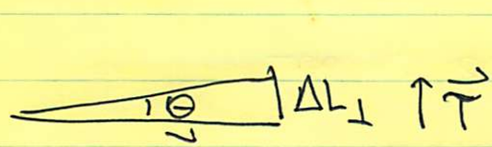
$$\tan\theta = \frac{\Delta p_\perp}{p}$$

~~$\Rightarrow \Delta p_\perp =$~~

The larger is $\vec{p}$, the smaller is θ
($\Delta p_\perp$ is fixed by the applied force)

→ fixed force is less successful in deflecting objects.

Rotational motion



$$\tan\theta = \frac{\Delta L_\perp}{L}$$

~~This~~ The larger $\vec{L}$ is, the harder it is the change direction by applying torque.
[guy ~~on~~ holding bicycle wheel → easy to flip when spinning slowly, hard to flip when spinning quickly].

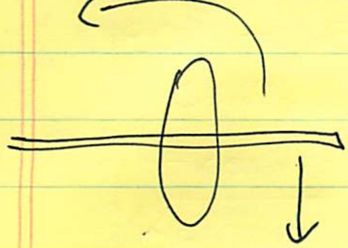
Examples

→ bicycle Angular momentum of wheels give stability, even though balanced on narrow tires.
large $\vec{L}$ → wheels will tend to fix orientation.

→ football  spin of football stabilizes it, prevents it fr. tumbling. This orientation also reduces air resistance.

→ satellite  given spin so ~~ca~~ that can maintain. orientation (otherwise buffeted around by air resistance fr. thin atmosphere, solar wind, or impacts fr. meteoroids).

Special section: more on gyroscopes



It seems crazy that when let go of bicycle wheel, instead of falling down, it goes sideways.

Actually, it ~~does~~ fall. If top is not spinning too fast, you can see it.

As soon as it falls,

it starts to

turn

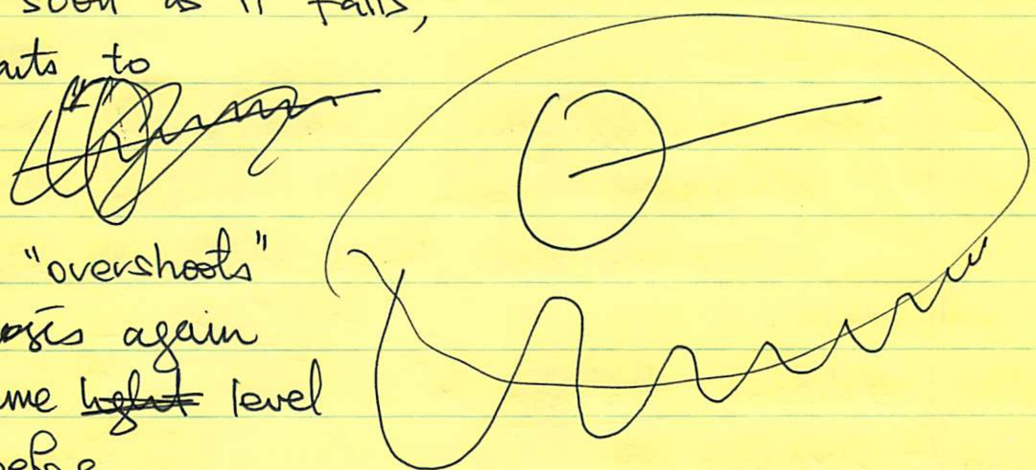
as in steady precession → but

motion "overshoots"

axis rises again

to same ~~height~~ level

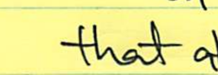
as before.



Trace out path of cycloid → NUTATION

show video of this

When motion settles down, axis a bit lower that at start



so than spin $\vec{L}$ has a small vertical cmpt, which is what is needed for precession

 ← if perfectly horizontal, then $\vec{L}$ has no vertical component.

Vector nature of $\vec{L}$ & $\vec{\omega}$

~~Main lessons~~

So far, we have always pursued a successful analogy between linear & angular momentum

$\vec{p} = m\vec{v} \rightarrow \vec{p}$ is alway llel to $\vec{v}$

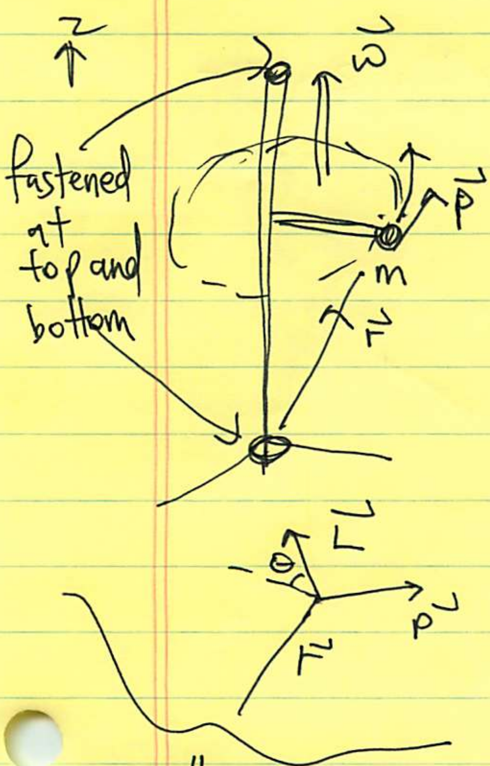
$\vec{L} = I\vec{\omega}$ is not always true ...

$\Downarrow$ in fact

so analogy breaks down... $\boxed{\text{~~But~~ } \vec{L} \text{ is not always parallel to } \vec{\omega}\text{ !}}$

(N.B. I is in general not a scalar or even a velocity, but a tensor, with 9 components!).

Let's ~~to~~ see this ...



Assume rods are rigid & massless. Ignore gravity. → only force is centripetal force.

$\vec{\omega}$ is in $\hat{z}$ direction → $\vec{\omega}$ is always $\perp$ to plane of rotation. Direction given by right hand rule.

But $\vec{L} = \vec{r} \times \vec{p}$ is not!

If origin is in plane of particle, then $\vec{L}$ is llel to $\vec{\omega}$, otherwise, it is not ...

$\Downarrow$

Show diagrams from HRK → Fig 6, Fig 7.

~~We care about~~

We want to use O as origin

because it is where the apparatus is attached.
If we don't also attach at A, the whole thing will wobble → ~~bearing~~ attachments have to exert torque to keep axis straight.

Contrast with case when have 2 equal & opposite masses ↘ (SHOW DIAGRAM).

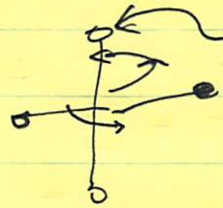
Then horizontal components are equal & opposite: cancel, so that $\vec{L}$ is only along $\hat{z}$ axis.

In this case, vector relation $\vec{L} = I\vec{\omega}$ is not true, since $\vec{L}$ & $\vec{\omega}$ point in different directions. In fact, only $L_z = I\omega$ is true.

~~If~~

If a rigid body is symmetric about axis of rotation, then there is always an equal & opposite mass element

So $\vec{L} = I\vec{\omega}$ for axial symmetry



can remove upper bearing and shaft will remain parallel.

~~Small bearing asymmetry → shaft will wobble~~

Small asymmetry in spinning object → shaft will wobble, requires torque on shaft (applied by bearings) to keep it stable.

Serious problem for objects that rotate at high speed → e.g. turbine rotors.
Spin in special device to measure wobble → correct by removing metal in right places.
"Balance" tires → add lead weight to reduce wobble at high speeds ⇒ make sure $\vec{L}$ & $\vec{\omega}$ are $\perp$!

<u>Important theorem</u>

(State, but cannot prove):

Any rigid body (even an irregular one) has 3 mutually perpendicular axes through the CM, s.t. moments of inertia about these axes

$$I_1 \leq I_2 \leq I_3$$

<u>least</u> possible value through c.m.

<u>greatest</u> possible value for any axis through CM

s.t. if body is rotating about them, then $\vec{L}$ is ∥ to $\vec{\omega}$.

~~For a body~~ These axes are known as the <u>principal axes</u>.

For body w/ axes of symmetry, the principal axes are along the symmetry axes.