Week 6: The Harmonic Oscillator

As I've mentioned, many things you study appear over and over again: e.g.,
propagation of sound waves $\rightleftarrows$ light waves.

This is another reason for studying mechanics: many of the themes you see will appear again!

Simple harmonic oscillator is an example of a <u>linear differential equation with constant coefficients</u>:

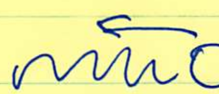
$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots a_1 \frac{dx}{dt} + a_0 x = f(t)$$

[linear ODE of order n].

Arises in <u>many</u> situations, for instance:
- oscillations of mass on spring
- oscillations of charge in electrical circuit.
- vibrations of tuning fork (sound)
- vibrations of electrons in atom (light)
- complicated chemical reactions
- thermostat
- ecology (growth of colony of bacteria .. foxes eating rabbits eating grass).

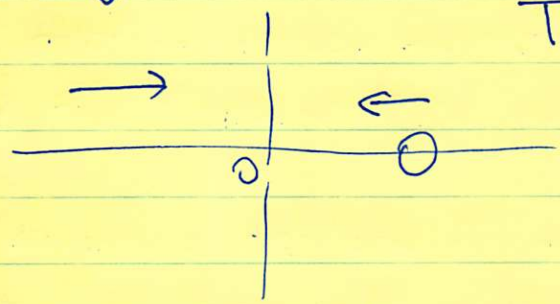
Harmonic oscillator

Pendulum, spring experience **restoring forces**, push back

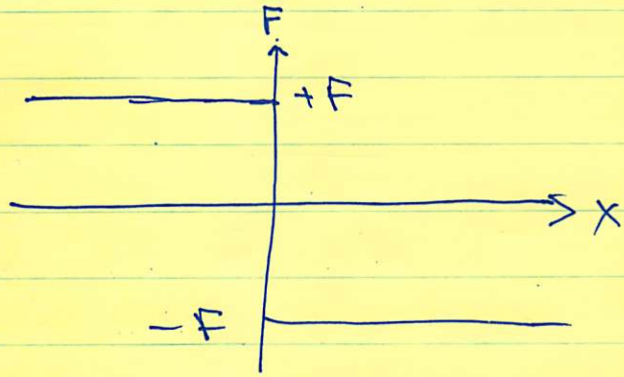


toward equilibrium.

Consider a simple example



This results in a **stable** equilibrium $\rightarrow$ system tends to return to equilibrium when slightly displaced (next time: do linear stability analysis).

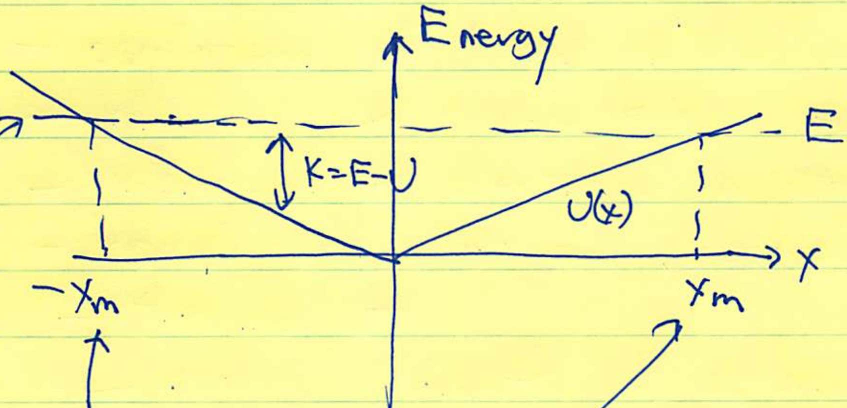


$$F = -\frac{dU}{dx} = \text{const}$$

$$\Rightarrow \cancel{0} \;\; \cancel{= A\ (x>0)} \;\; \cancel{-A\ (x<0)}$$

$$= +F \quad (x<0)$$

$$-F \quad (x>0)$$



TOTAL energy is conserved

places where K.E. = 0

$\Rightarrow$ **turning points** of system.

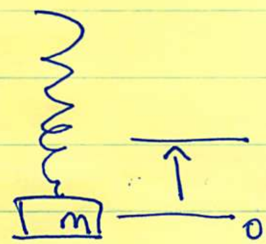
$$\Rightarrow U = -Fx \quad (x<0)$$

$$Fx \quad (x>0)$$

~~$F = ma$~~ Conditions for oscillation:
~~$\Rightarrow F =$~~ ① Forces act to restore
② potential energy must have a minimum at equilibrium point (so $F = -\frac{dU}{dx} = 0$ ~~there~~

→ unperturbed mass will sit at equilibrium point).

Simple harmonic oscillator



Consider mass on spring, and force ~~wh~~ which is linear w/ displacement

$$F = -kx.$$

pulls in opposite direction fr. stretch

$$= ma$$
$$= m\ddot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + kx = 0}$$

You already know the solution to this equation.
But let's ~~lo~~ look at this more carefully.
Consider first

$$\boxed{\ddot{x} = -x}$$

→ want function which equals itself when differentiated 2x, with minus sign.

$x = \cos t$	$x = \sin t$
$\dot{x} = -\sin t$	$\dot{x} = \cos t$
$\ddot{x} = -\cos t$ ✓	$\ddot{x} = -\sin t$ ✓

Now how to solve

$$m\ddot{x} = -kx.$$

Maybe can do it by multiplying cos t w/ sth.

try

$$x = A\cos t$$

$$\dot{x} = -A\sin t$$

$$\ddot{x} = -A\cos t = -x$$ ~~same~~ solution to $\ddot{x} = -x$ instead!

Important property of linear ODE
→ if multiply sol$^{\underline{n}}$ by a const, it's again a sol$^{\underline{n}}$.
(Physically: — if pull spring 2x as far, then acceleration, velocity 2x larger but has 2x dist to cover :→ takes same time to get back.
Similarly, pendulum period is independent of amplitude).

So the characteristic of the eq$^{\underline{n}}$ is the pattern in <u>time</u>, not the scale of <u>x</u>.

Thus, to solve $m\ddot{x} = -kx$, we must change the scale of time

Try:

$$x = \cos \omega_0 t$$

just ~~to remind ourselves~~

Note, this is angular <u>frequency</u>, not angular velocity of rotating body.

Then $\dot{x} = -\omega \sin \omega t$

$\ddot{x} = -\omega^2 \cos \omega t = -\omega^2 x$

$$m\ddot{x} = -kx \Rightarrow \ddot{x} = -\frac{k}{m} x \Rightarrow \boxed{\omega^2 = \frac{k}{m}}.$$

What is physical significance of ω?
cos repeat every 2π, i.e. when $\omega T = 2\pi$

$$\Rightarrow \text{period } T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}.$$

heavier mass, longer period
stronger spring, shorter period.

Does not depend on amplitude of motion.

~~Most general sol~~

Since independent of amplitude, & both sin & cos are solutions, the most general sol$^{\underline{n}}$ is

$$x = A \cos \omega t + B \sin \omega t.$$

This also covers the case when we shift the origin of time,

suppose have $x = a \cos \omega t$ as sol$^{\underline{n}}$

but $x = a \cos \omega (t - t_1)$ is also possible (shift origin of time).

$$\Rightarrow x = a \cos \omega t + \Delta$$

$$= a[\cos \omega t \cos \Delta - \sin \omega t \sin \Delta]$$

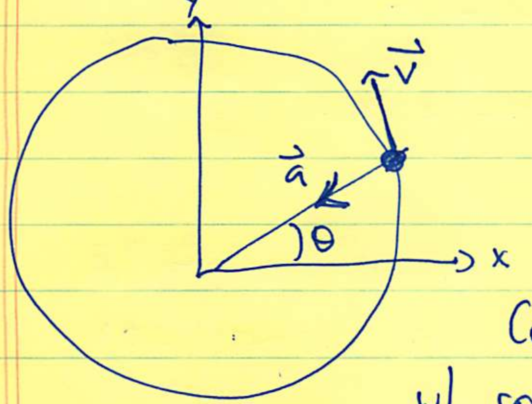
$$= A \cos \omega t + B \sin \omega t$$

$A = a \cos \Delta$
$B = -a \sin \Delta$.

a = max displacement or Amplitude

Δ = phase shift.

Harmonic motion & circular motion



We've actually already solved for SHO before ... when studying circular motion.

Consider a particle moving in a circle w/ const speed v.

$$\left.\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned}\right\} \; x^2 + y^2 = R^2$$

But it also ~~express~~ experiences an acceleration

$$\begin{aligned} a_x &= -a\cos\theta \\ &= -\left(\frac{v^2}{R}\right)\cos\theta = -\frac{(\omega R)^2}{R}\cos\theta \\ &= -\omega^2 R\cos\theta \\ &= -\omega^2 x. \end{aligned}$$

~~x = R cos θ. SHM~~

Similarly for y.

~~SHM~~ SHO can be described as the projection of uniform circular motion along the diameter of a circle.

When Galileo first measured the orbits of moons about Jupiter, he got motion which was SHO.

Can regard ~~SHO as~~ circular motion as combination of SHO at $\perp$, with identical frequency & amplitude, but differing in phase by 90°.

Initial conditions

$$x = A\cos\omega t + B\cancel{\cos\omega t}\sin\omega t$$

What determines A & B?

They are determined by <u>initial conditions</u>.

$$\dot{x} = v = -\omega A\sin\omega t + \cancel{\omega B\sin}\ \omega B\cos\omega t.$$

$$\text{At } t=0, \quad \left.\begin{aligned} x_0 &= A \\ v_0 &= \omega B \end{aligned}\right) \text{ ~~How~~ determine } A, B.$$

Conservation of energy $\rightarrow$ since there are no frictional losses, energy should be conserved.

$\cancel{U = \frac{1}{2}kx^2}$ Let's use

$$x = a\cos(\omega t + \Delta)$$

$$v = -\omega a\sin(\omega t + \Delta)$$

then

$$U = \frac{1}{2}kx^2 = \frac{1}{2}ka^2\cos^2(\omega t + \Delta)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 a^2\sin^2(\omega t + \Delta)$$

Note that K. E. & P.E. are 90° out of phase: KE is max when PE is min, & vice-versa

Use $\omega^2 = \frac{k}{m} \Rightarrow k = \omega^2 m$

$$\Rightarrow E = T + U = \frac{1}{2} m\omega^2 a^2 \left(\cos^2(\omega t + \Delta) + \sin^2(\omega t + \Delta)\right]$$

$$= \frac{1}{2} m\omega^2 a^2$$

$$= \frac{1}{2} k a^2.$$

$\underline{E \propto a^2} \Rightarrow$ ~~twice~~ $a \to 2a$, $E \to 4a$.

$$\langle U \rangle = \cancel{\frac{1}{2}} \left\langle \frac{1}{2} m\omega^2 a^2 (\cos^2(\omega t + \Delta) \right\rangle$$

$$= \frac{1}{4} m\omega^2 a^2 = \frac{1}{2} E$$

similarly $\langle T \rangle = \frac{1}{2} E$.

Combination of Harmonic Motion

Consider 2SHO at right angles; sum of 2 independent oscillations.

$$x = x_m \cos(\omega t + \phi_x)$$
$$y = y_m \cos(\omega t + \phi_y)$$

$\Leftarrow$ first consider case when frequencies are the same.

Phase constants same $(\phi_x = \phi_y)$: $\longrightarrow$ $\frac{x}{y} = \frac{x_m}{y_m}$

~~x, y are in phase~~

$$\Rightarrow \boxed{y = \left(\frac{y_m}{x_m}\right) x}$$ straight line!

x, y are <u>in phase</u>: they reach their maxima and their minima at the same time.

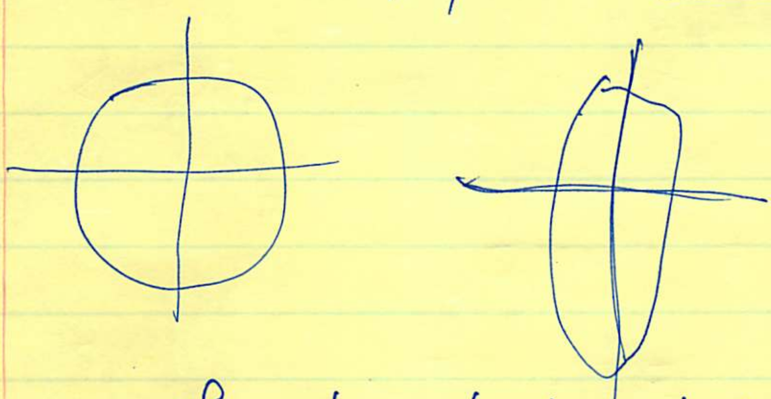
$\phi_x \neq \phi_y$

E.g., $\phi_x = \phi_y + \pi/2$

Then max x-displacement when min y displacement & vice versa.

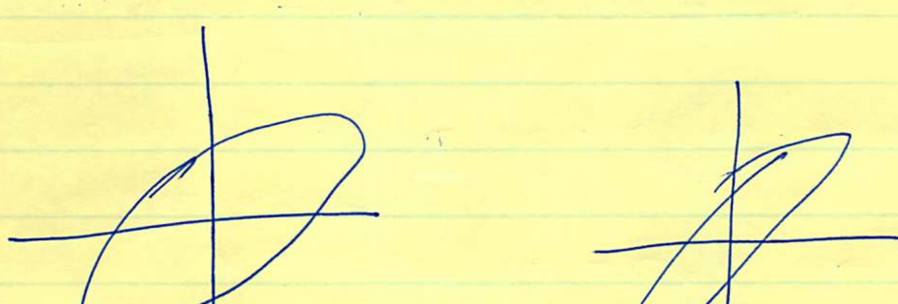
$x_m = y_m \Rightarrow$ circular

$x_m \neq y_m \Rightarrow$ elliptical.



$y_m = 2x_m$.

If $\phi_x - \phi_y$ ~~has~~ $\neq \pi/2$, get tilted ellipse.



This all comes fr. fact that straight line, circle are all special cases of ellipse:

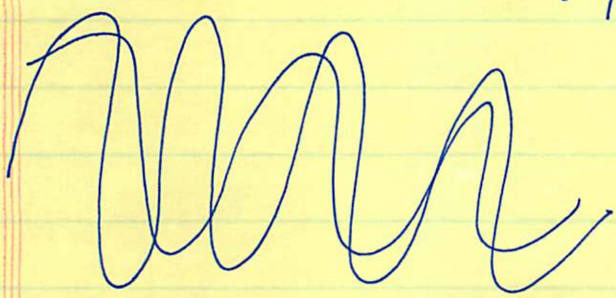
$$\cancel{x = x_m \cos(\omega t + \phi_x)}$$
$$\cancel{y = y_m \cos(\omega t + \phi_y)}$$

Depends only on ratios of amplitude $\frac{y_m}{x_m}$ & difference in phase $\phi_x - \phi_y$

$$\cancel{\Rightarrow \omega t + \phi_x = \cos^{-1}\left(\frac{x}{x_m}\right)}$$
$$\cancel{\Rightarrow t = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{x}{x_m}\right) - \phi_x\right] = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{y}{y_m}\right) - \phi_y\right]}$$

Can be clockwise or counterclockwise, depending on which component leads in phase ..

2 different frequencies : more complicated. Motion not periodic unless $\frac{\omega_x}{\omega_y} = \frac{m}{n}$, where m, n are integers.



← e.g., Lissajous figure on oscilloscope.

We've only considered SHO in different directions (at $\perp$ to one another). Consider combinations of SHO in same direction, but w/ different amplitudes & phases → diffraction, interference seen in light, sound, QM ....

DAMPED HARMONIC MOTION

If no frictional forces act, pendulum/mass on string oscillates indefinitely.
In reality, damped by friction — damped harmonic motion

$$F_{drag} = -bv$$

$$\Rightarrow \quad m\ddot{x} = F_{spring} + F_{drag}$$

$$= -kx - bv$$

$$\Rightarrow \quad m\ddot{x} + b\dot{x} + kx = 0.$$

Solution is:

$$x = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$$

Check:

$$\dot{x} = -\frac{b}{2m}\overbrace{\left[x_m e^{-\frac{bt}{2m}}\cos(\omega' t+\phi)\right]}^{x}$$

$$- \omega' x_m e^{-\frac{bt}{2m}} \sin(\omega' t+\phi)$$

$$\ddot{x} = -\frac{b}{2m}\dot{x} + \frac{\omega' b}{2m} x_m e^{-\frac{bt}{2m}}\sin(\omega't+\phi)$$

$$- \omega'^2 x$$

$$= \left[\left(\frac{b}{2m}\right)^2 - \omega'^2\right]x + \left[\frac{\omega' b}{\cancel{2}m}\right] x_m e^{-\frac{bt}{2m}}\sin(\omega't+\phi)$$

$$m\ddot{x} + b\dot{x} + kx = 0 \Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \cancel{m\left[\left(\frac{b}{2m}\right)^2 - \omega'^2\right]}$$

$$\Rightarrow \left[\left(\frac{b}{2m}\right)^2 - \omega'^2 - \frac{b^2}{2m^2} + \frac{k}{m}\right] x + (\text{sin part cancels}) = 0$$

$$\Rightarrow \omega'^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2 \Rightarrow \omega = \underbrace{\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}$$

require this to be > 0.

Can actually show sol$\underline{n}$ more easily:

$$m\ddot{x} + b\dot{x} + kx = 0$$

Try

$$x = \cancel{e^{-t}}\, e^{\omega t}$$

$$\Rightarrow \quad m\omega^2 + b\omega + k = 0$$

$$\Rightarrow \quad \omega = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$$\Rightarrow \quad \omega = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

$$\cancel{= e^{-\frac{b}{2m}t}}$$

$$= -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

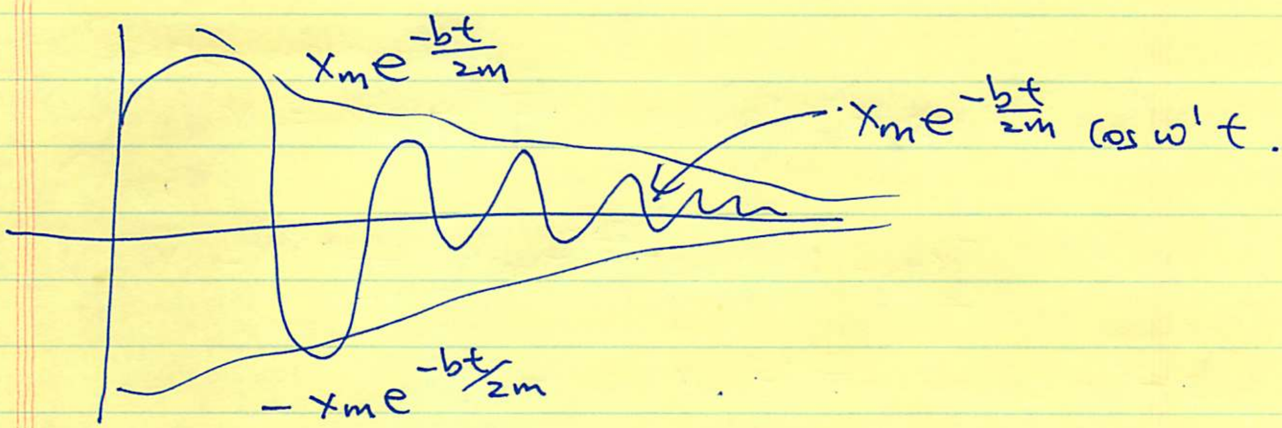
$$\Rightarrow \quad x = x_m\left[e^{-\frac{b}{2m}t}\, e^{i\tilde{\omega}t}\right]$$

$$\tilde{\omega} = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

$\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ > 0 underdamping

$= 0$ critically damped

< 0 overdamped.



① $\omega' < \omega \Rightarrow$ friction slows down motion, lower frequency.

② Amplitude of motion decays exponentially.

$$e^{-bt/2m} = e^{-t/\tau} \Rightarrow \tau = \frac{2m}{b} \text{ is the mean lifetime.}$$

$$E \simeq \frac{1}{2} k a^2$$

a ← instantaneous value of amplitude

$$= \frac{1}{2} k \left(x_m e^{-\frac{bt}{2m}} \right)^2$$

$$= \cancel{\tfrac{1}{2}k} = E e^{-\frac{bt}{m}}$$

energy decays exponentially.

N.B. solutions only valid for $\cancel{b \leq 2\sqrt{km}}$

$\cancel{\omega' > 0}$

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \geq 0$$

$$\Rightarrow \quad b \leq 2\sqrt{km}$$

For

$b = 2\sqrt{km}$, $\omega' = 0 \Longrightarrow$ displacement decays w/ no oscillation.

$$\tau = \frac{2m}{b} = \frac{2m}{2\sqrt{km}} = \sqrt{\frac{m}{k}} = \frac{1}{\omega}$$

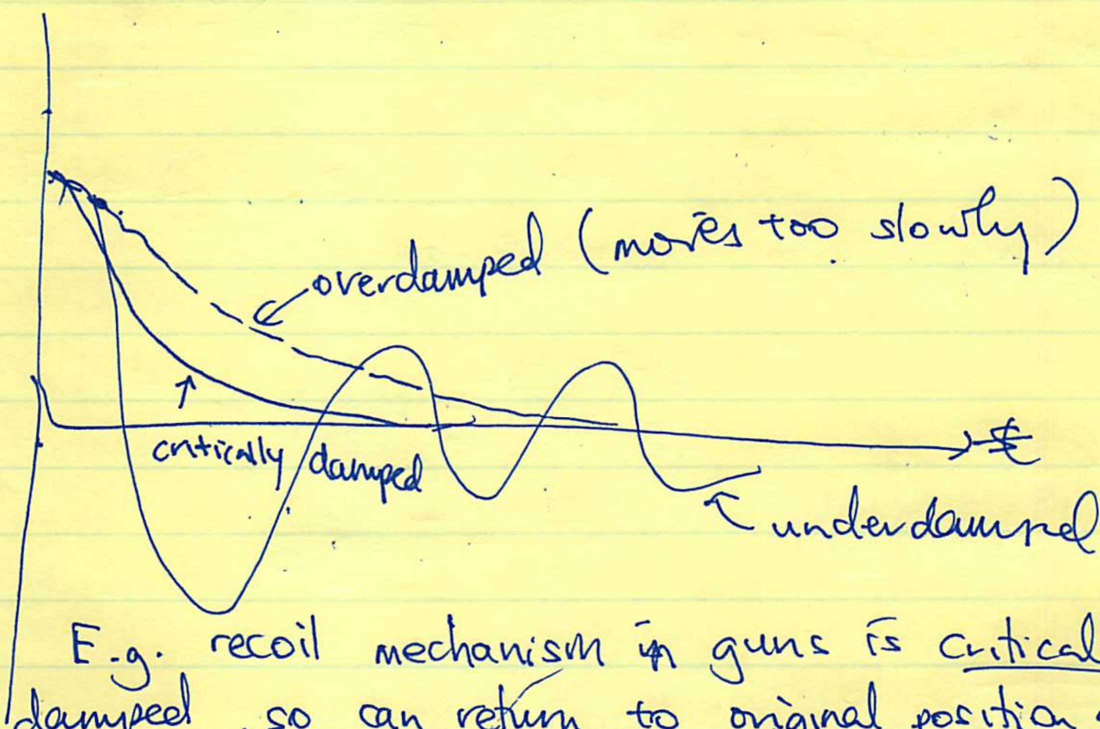
smallest value

→ critical damping

Goal of mechanical engineer in designing system where oscillations disappear in shortest possible time.

What if $b > 2\sqrt{km}$:

System is overdamped



E.g. recoil mechanism in guns is critically damped, so can return to original position after recoil due to firing in least possible time.

Forced oscillations

Suppose a body is subject to an external driving force.
e.g. → bridge vibrates due to marching soldiers
→ eardrum vibrates due to sound wave.

<u>Forced</u> oscillation have freq. of external force, not the natural frequency of the body.

Consider the case where there is no damping:

$$m\ddot{x} = -kx + F(t)$$

Let $F = F_m \cos \omega'' t$.

Let's try as a solution

$$x = C \cos \omega'' t.$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\Rightarrow k = m\omega_0^2$$

Then $-m\omega''^2 C \cos \omega'' t \; \cancel{= k\omega_0^2}$

$$= -m\omega_0^2 C \cos \omega'' t + F_0 \cos \omega'' t$$

$$\Rightarrow \quad C = \frac{F_0}{m(\omega_0^2 - \omega''^2)}$$

So m oscillates at frequency of force, but <u>amplitude</u> depends on $\omega_0^2 - \omega''^2$

natural frequency

frequency of force

$\omega'' \ll \omega_0 \rightarrow C > 0 \Rightarrow$ displacement & force in same direction

$\omega'' > \omega_0 \Rightarrow C < 0 \Rightarrow$ displacement & force are out of phase.

Note

$|\omega_0^2 - \omega^2| \gg \frac{F_0}{m} \Rightarrow C \ll 1$

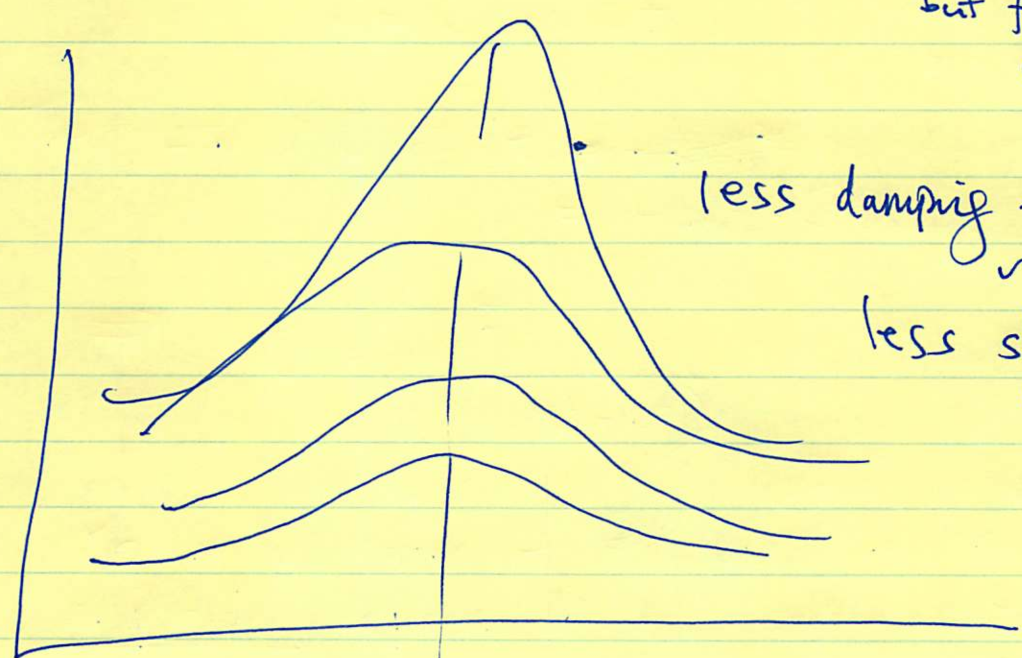
$\Rightarrow$ amplitude of oscillations small.

This is the steady-state response.
In practice, there is also a transient response.

As $\omega^2 \rightarrow \omega_0^2$ ("resonance")

$C \rightarrow \infty$. $\Rightarrow$ in practice, friction, etc means a large but finite amplitude.

Consider child on swing!



less damping – sharper resonance.
less shift fr. natural frequency.

E.g. – soprano shattering wine glass
– Tacoma Narrows Bridge disaster.