Simple Harmonic Oscillator : K & K

You already learnt a good deal of K & K Chap 10 from HRK!

These lectures will "fill in the blanks" so that you are up to speed to tackle some cool K & K problems.

~~I~~ SHO : Standard Form of Solution

Consider $\ddot{x} = -kx$

I told you the standard form of the solution is:

$$x = B \sin \omega_0 t + C \cos \omega_0 t.$$

where B, C are constants.

This can be more ~~conviently~~ convienently rewritten as:

$$x = A \cos(\omega_0 t + \phi)$$

where A, ϕ are constants (still 2 degrees of freedom).

We can show this by relating the old & new soln:

$$x = A\cos(\omega_0 t + \phi)$$
$$= A\left[\cos\omega_0 t \cos\phi - \sin\omega_0 t \sin\phi\right)$$

But $x = B \sin\omega_0 t + C\cos\omega_0 t$

$$\Rightarrow B = -A\sin\phi \quad C = A\cos\phi.$$

or $B^2 + C^2 = A^2(\sin^2\phi + \cos^2\phi) = A^2$

$$\Rightarrow A = \sqrt{B^2 + C^2}$$

$$\tan\phi = -\frac{B}{C}$$

We will use $x = A\cos(\omega_0 t + \phi)$ as a more convenient & compact form

amplitude (A), phase (ϕ).

$\boxed{\text{DAMPED OSCILLATORS}}$

Q of an Oscillator

Q → the quality factor

Q is a dimensionless parameter for a damped oscillator which specifies the degree of damping.

$$Q \equiv \frac{\text{energy stored in oscillator}}{\text{energy dissipated per radian}}$$

Why 1 radian?

In a period $T = \frac{2\pi}{\omega}$, the system oscillates through 2π radians.

Hence time to oscillate through 1 radian is

$$t = \frac{T}{2\pi} = \frac{1}{\omega}$$, much more convenient.

For the lightly damped case, easy to calculate Q.

Recall $E = E_0 e^{-\gamma t}$ $\quad \gamma = \frac{b}{m}$

$\frac{dE}{dt} = -\gamma E_0 e^{-\gamma t}$ $\rightarrow$ energy falls exponentially

$= -\gamma E.$

In a short time Δt,

$$\Delta E = \left|\frac{dE}{dt}\right| \Delta t$$

$$\simeq \gamma E \Delta t$$

$$= \frac{\gamma E}{\omega_1}$$

$\Delta t = \frac{1}{\omega_1}$ for one radian of oscillation

Hence

$$Q = \frac{E}{\Delta E} = \frac{E}{\frac{\gamma E}{\omega_1}} = \frac{\omega_1}{\gamma} \simeq \frac{\omega_0}{\gamma}$$

frequency of damped oscillator

Lightly damped $Q \gg 1$ e.g. superconducting microwave cavity $Q > 10^7$

Heavily damped low Q tuning fork $Q \sim 10^3$.

Q of Tuning Fork

Musician's tuning fork rings at $\nu_0 = 440$ Hz (A above middle C).

Fr. sound meter, intensity decreases by a factor of 5 in 4s.

$$\frac{E_0 e^0}{E_0 e^{-4\gamma}} = 5 \Rightarrow e^{-4\gamma} = \frac{1}{5}$$

$$\Rightarrow \ln e^{-4\gamma} = -\ln 5$$

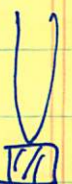
$$\Rightarrow \gamma = \frac{1}{4}\overset{1.6}{\ln 5} = 0.4\ s^{-1}.$$

$$Q = \frac{\omega_1}{\gamma} = \frac{2\pi\nu}{\gamma} = \frac{2\pi(440)}{0.4} = 7000$$

Energy loss mainly due to heating of metal as it bends (also: air friction, energy loss to mount).

Rubber band Q

[NOTE: ear is poor sound meter because response is logarithmic not linear (just like eye!)]



Consider weight suspended from rubber band.
Suppose period $T = 1.2s$.
Amplitude A decreased by a factor of 2 after 3s.

What is the estimated Q?

$$E \propto A^2 \propto e^{-\gamma t} \Rightarrow A \propto e^{-\frac{\gamma t}{2}}.$$

$$A = A_0 e^{-\frac{\gamma t}{2}}.$$

$t = 0 \qquad A = A_0$

$t = 3(1.2)$
$= 3.6s \qquad A = A_0 e^{-\frac{3.6\gamma}{2}} = \frac{A_0}{2}$

$$\Rightarrow \ln e^{-\overset{1.8}{\cancel{3.6}}\gamma} = \ln\frac{1}{2}$$

$$\Rightarrow \gamma = \frac{1}{1.8}\overset{0.69}{\ln 2}$$

$$= 0.39\ s^{-1}.$$

$$\Rightarrow Q = \frac{\omega_1}{\gamma} = \frac{2\pi/T}{0.39} = \frac{2\pi/1.2}{0.39} = 13$$

[N.B., we are using light damping result, which is accurate to $O\left(\frac{1}{Q^2}\right)$. For $Q > 10$, error is $< 1\%$].

γ for rubber band & tuning fork are ~~very diff~~ very similar
→ but Q is very different, as natural frequencies ω_1 are v. different. (so rubber band loses more energy per cycle).

Losses for rubber band come fr. internal friction [coiling & uncoiling of long chain molecules].

Graphical Analysis of Damped Oscillator
[LATER... if there's time].

Forced Harmonic Oscillator

Undamped case:

$$m\ddot{x} + kx = F_0 \cos \omega t .$$

Why choose a periodic driving force?

1. Fourier's theorem = any periodic function of time can be represented as a sum of sinusoidal terms

$$f(t) = \sum_n a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

2. Many important cases have a sinusoidal driving force:
   i) tides
   ii) response of bound electron to EM field (happens in scattering of light).

Recall we found the solution

$$x = \frac{F_0}{m} \frac{1}{\omega_0^2 - \omega^2} \cos \omega t .$$

But this is weird! Recall our other solutions:

$$x = A \cos(\omega_0 t + \phi) \quad \text{SHO}$$
$$x = A e^{-(\gamma/2 t)} \cos(\omega_1 t + \phi) \quad \text{damped HO}.$$

These all have 2 undetermined constants, A, ϕ, which we use to match x, v at the initial time.

But the solution we've found has no arbitrary constants! So how do we match initial x & v?

In fact, the complete solution is

$$x = \underbrace{\frac{F_0}{m}\frac{1}{\omega_0^2-\omega^2}\cos\omega t}_{\text{particular solution } \cancel{\text{to partic}} \text{ to } \ddot{x}+kx=F(t)} + \underbrace{B\cos(\omega_0 t+\phi)}_{\text{general sol}^{\underline{n}}\text{ of homogeneous eq}^{\underline{n}}\ \ddot{x}+kx=0.}$$

complete sol$^{\underline{n}}$ is sum of homogeneous & particular solution.

B, ϕ are arbitrary; can be used to match initial conditions.

If system is damped, then ~~soln~~ B decreases exponentially with time → can be ignored. We are left with

$$x = \frac{F_0}{m}\frac{1}{\omega_0^2-\omega^2}\cos\omega t,$$

the <u>steady-state</u> solution.

There is a phase change of 180° through resonance!

$\omega < \omega_0 \longrightarrow$ displacement in phase w/ driving force

$\omega > \omega_0 \longrightarrow$ displacement 180° out of phase w/ driving force [when force is max, displacement is min & vice versa].

Resonance → can use it to get large response from a small driving force: i) organ pipes.
ii) radio (resonant electrical circuits).

But also want to damp resonance sometime (springs of car)
↳ shock absorber.

$\boxed{\text{Forced Damped Harmonic Oscillator}}$ ← new! Not covered in HRK.

$$F = m\ddot{x} = F_{spring} + F_{damp} + F_{drive}$$

$$= -kx - bv + F_0 \cos \omega t$$

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t.$$

$$\Rightarrow \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

$$\gamma = \frac{b}{m}, \quad \omega_0^2 = \frac{k}{m}.$$

Try $x = A\cos(\omega t + \phi)$ as solution.

Then

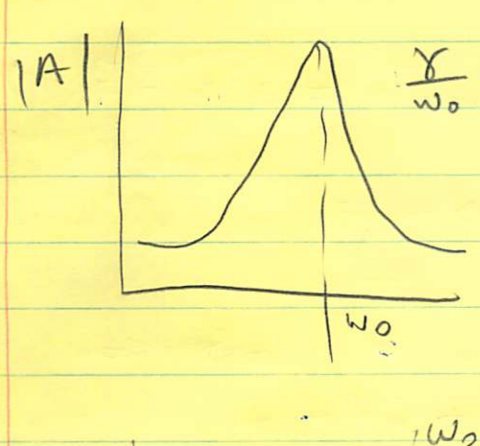
$\cancel{-\omega^2 x}$

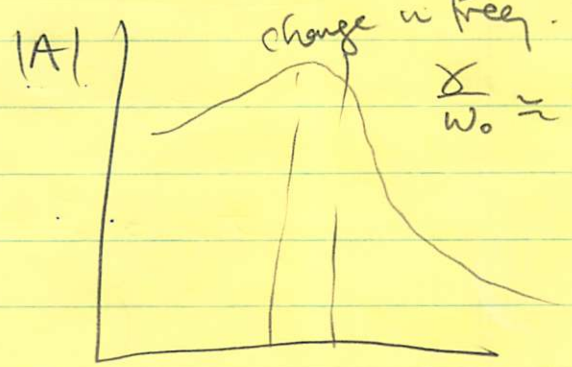
$$-\omega^2 A\cos(\omega t+\phi) - \gamma A\omega\sin(\omega t+\phi) + \omega_0^2 A\cos(\omega t+\phi) = \frac{F_0}{m}\cos\omega t$$

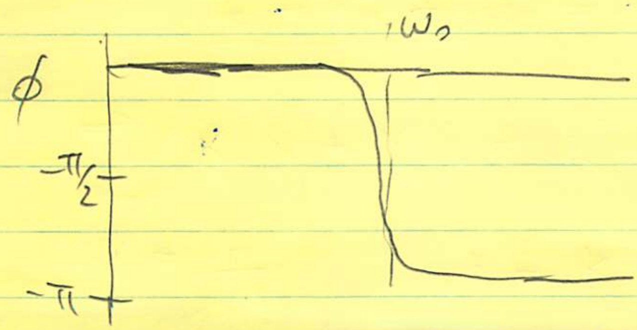
$$\Rightarrow A\cos(\omega t+\phi)\left[-\omega^2 - \gamma\omega\tan(\omega t+\phi) + \omega_0^2\right] = \frac{F_0}{m}\cos\omega t$$

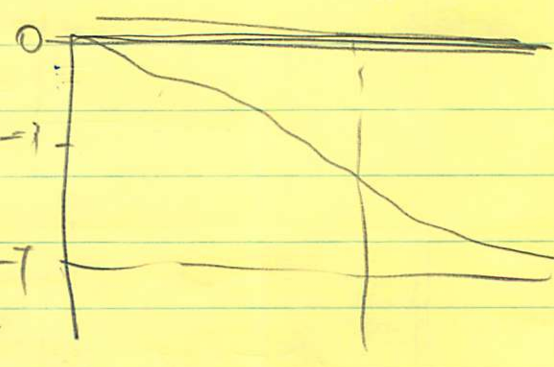
$$A = \frac{F_0}{m}\frac{1}{\left[(\omega_0^2-\omega^2)^2+(\omega\gamma)^2\right]^{1/2}}$$

$$\phi = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right) \text{ is solution.}$$









As $\gamma \to 0$, phase change occurs more abruptly

Resonance in Lightly Damped Forced Oscillator

Let's calculate the quality factor Q here
For steady-state motion, amplitude is const in time.

$$x = A\cos(\omega t + \phi)$$
$$\dot{x} = -A\omega\sin(\omega t + \phi)$$

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi)$$
$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}k^2A^2\cos^2(\omega t + \phi)$$

$$E(t) = K(t) + U(t)$$
$$= \frac{1}{2}A^2\left[m\omega^2\sin^2(\omega t + \phi) + \underbrace{k}_{m\omega_0^2}\cos^2(\omega t + \phi)\right]$$

$$\cancel{= \frac{1}{2}Am\omega^2}$$

$$\langle K(t)\rangle = \frac{1}{2}A^2m\omega^2\langle\sin^2(\omega t + \phi)\rangle$$
$$= \frac{1}{4}Am\omega^2$$

$$\langle U(t)\rangle = \frac{1}{2}A^2m\omega_0^2\,\langle\cos^2(\omega t + \phi)\rangle$$
$$= \frac{1}{4}Am\omega_0^2$$

$$\langle E(t)\rangle = \frac{1}{4}mA(\omega^2 + \omega_0^2)$$

$$\langle E(t)\rangle = \frac{1}{4}\frac{F_0^2}{m^{\cancel{2}}}\frac{1}{[\omega_0^2-\omega^2]^2+(\omega\gamma)^2}\cancel{m}(\omega^2+\omega_0^2)$$

Consider light damping, $\gamma \ll \omega_0$.

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \simeq \omega_0$$

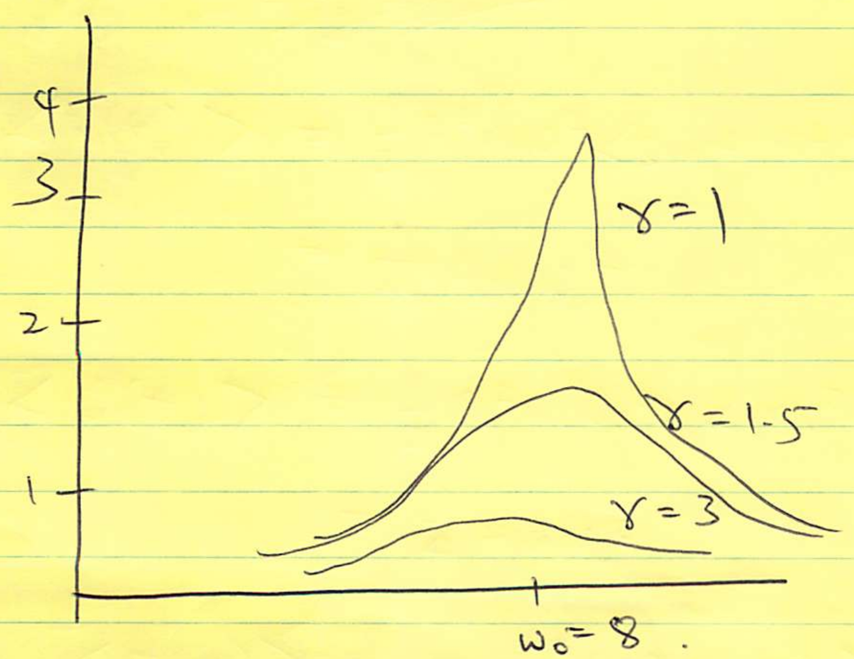
Except for $(\omega_0^2-\omega^2) \simeq (\omega_0+\omega)(\omega_0-\omega)$

$$= 2\omega_0(\omega_0-\omega)$$

$$\Rightarrow \langle E(t)\rangle = \frac{1}{4}\frac{F_0^2}{m}\frac{2\omega_0^2}{\omega_0^2\gamma^2+4\omega_0^2(\omega-\omega_0)^2}$$

$$= \frac{1}{8}\frac{F_0^2}{m}\left(\frac{1}{(\omega-\omega_0)^2+(\gamma/2)^2}\right)$$

Resonance curve, or <u>Lorentzian</u>



Consider Lorentzian

$$L = \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}.$$

L_{max} when $\omega = \omega_0 \Rightarrow L_{max} = \frac{4}{\gamma^2}$

$$L = \frac{1}{2} L_{max} = \frac{2}{\gamma^2} = \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}$$

$$\Rightarrow (\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2 = \frac{\gamma^2}{2}$$

$$\Rightarrow (\omega-\omega_0)^2 = \frac{\gamma^2}{4} = \left(\frac{\gamma}{2}\right)^2$$

$$\Rightarrow \omega - \omega_0 = \pm \gamma/2.$$

~~FWH~~ Full width at half max

$$\boxed{\Delta\omega = 2(\omega-\omega_0) = \gamma}$$

↳ As γ decreases, $\Delta\omega$ falls and resonance becomes sharper in frequency.

This sharpness in frequency is equivalent to the quality factor Q =

$$Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega} = \frac{\text{resonance freq.}}{\text{frequency width of resonance curve}}$$

for lightly damped oscillator.

This is the main use of Q $\rightarrow$ even though it's defined in terms of energy, mainly used to characterize the frequency response of a system.

Higher Q $\rightarrow$ more selective in frequency response.

Some atomic systems have $Q > 10^8$!

$\rightarrow$ only respond unless driven ω. close to resonant frequency (which is determined by fundamental consts)

$\rightarrow$ used as "atomic clocks".

RESPONSE IN TIME vs RESPONSE IN FREQUENCY

oscillator $\longrightarrow$ damping is decreased...

$\rightarrow$ energy more slowly dissipated

$\rightarrow$ becomes more frequency selective [$Q \uparrow$].

These 2 aspects are related!

Energy of damped oscillator w/out driving.

$$E = E_0 e^{-\gamma t} \qquad \text{damping time } \boxed{\tau = \frac{1}{\gamma}}$$

Suppose now drive w/ ~~force~~ force $F_0 \cos \omega t$.

Then $\boxed{\Delta\omega = \gamma}$

$$\Rightarrow \quad \Delta\omega = \tau^{-1} \Rightarrow (\Delta\omega)(\tau) = 1.$$

Cannot desegin oscillator where $\Delta\omega$, τ are independently specified $\rightarrow$ one determines the other!

Consequences $\rightarrow$ if γ is low

$\hookrightarrow$ will be highly frequency selective
and will oscillate for long time
if perturbed $\rightarrow$ take long time to reach steady state when driving force is applied.

Fundamental in quantum mechanics
Related to Heisenberg uncertainty principle