Fluid Dynamics

Lagrangian mechanics :— follow fluid particles with time, apply laws of motion to each of them.

Eulerian mechanics :— specify quantities at fixed positions

$$\left.\begin{array}{l}\rho(x,y,z,t)\\ v(x,y,z,t)\end{array}\right]$$ Focus on what's happening at a given point in space & time, rather than what's happening to a fluid particle.

Can have many associated properties :

① Steady or non-steady   Steady : $\rho(x,y,z,t) = \rho(x,y,z)$
$v(x,y,z,t) = v(x,y,z)$

Flow can ~~be~~ vary as a function of position, but not time.

E.g. in turbulent flows, velocities vary erratically fr. point to point as well as time to time.

② Compressible or incompressible Depends on conditions of flow — even if highly compressible, may behave like an incompressible fluid → e.g., subsonic flow over aircraft wings is nearly incompressible.

③ Viscous or non-viscous Analog of friction in fluid flow [N.B. motor oils are rated by viscosity, & variation w/ temperature].

④ Rotational or irrotational [when learn vector calculus → $\nabla \times \vec{v} = 0$].

Helmholtz theorem = every vector field can be written as

$$\vec{v} = -\nabla \phi - \nabla \times \vec{A}$$

scalar potential (ϕ), vector potential ($\vec{A}$).

If $\nabla \times \vec{v} = 0 \Rightarrow v = -\nabla \phi$, potential flow.

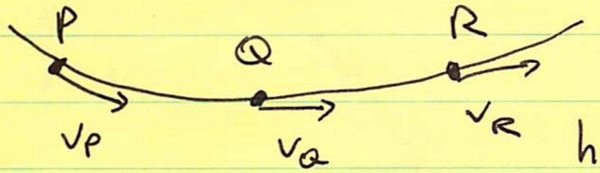
If element of moving fluid does not ~~move~~ rotate about its c.m., it is irrotational.

Does a free-floating paddle wheel rotate?
Yes → rotational
No → irrotational.

It's possible to move in a circular path & still be irrotational
→ cars in Ferris wheel [they don't rotate about their c.m.]
→ vortex about bathtub drain.

We are going to specialize to steady, incompressible, non-viscous, irrotational flow → i.e., assume these properties (viscosity, compressibility, etc) are ~~negle~~ negligible. ↳ "Consider a spherical cow..."

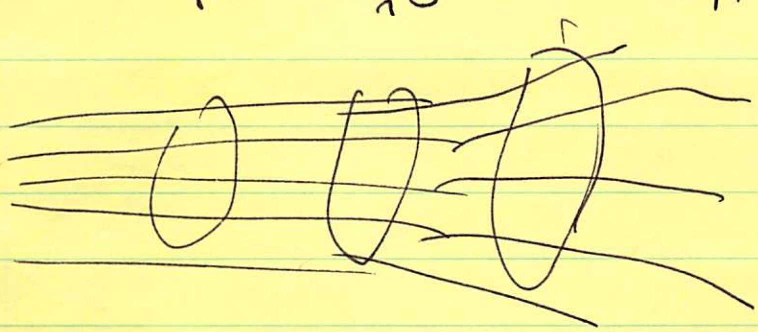
Streamlines and the Equation of Continuity



In steady flow, every particle passing through P, Q, R has same velocities and follows the same path, called a <u>streamline</u> [the "road" that particles follow].

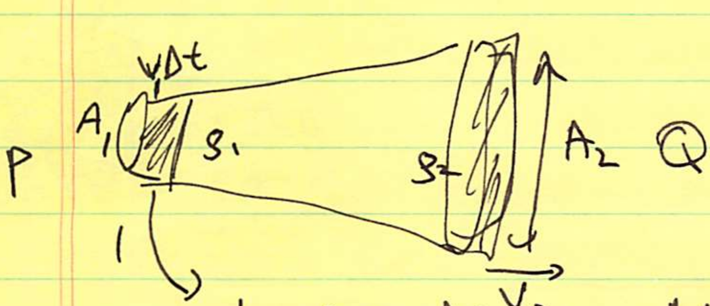
— note that velocity can change along a streamline (it's always tangent to streamline].

— streamlines <u>cannot</u> cross [otherwise fluid particle could go either way, and flow is not steady].



— ~~conne~~ consider a bundle of streamlines: <u>tube of flow</u>. Velocities are tangent to outer streamlines, cannot cross them

↳ acts like a pipe.



In time Δt,

$$\Delta M = \rho \Delta V = \rho A v \Delta t$$

$$\Rightarrow \dot{M}_1 = \rho_1 A_1 v_1$$

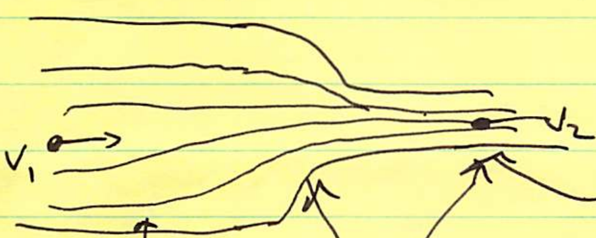
mass flux at Q $\quad \dot{M} = \rho_2 A_2 v_2$.

Flow is steady, no sources or sinks → mass flux can't change:

$$\rightarrow \rho_1 A_1 v_1 = \rho_2 A_2 v_2.$$

Or more generally, $\rho A v = \text{const}$. [conservation of mass.].

If incompressible, then $Av = \text{const}$ (vol flux is const).



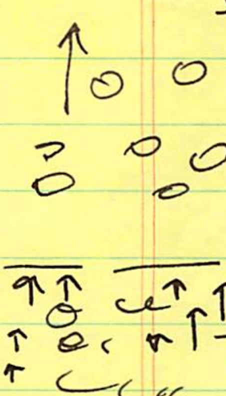
streamlines close together → flow quickly

widely seperated streamlines → flow slowly

$v_1 < v_2$

particle must be accelerated by a pressure gradient here

so regions w/ higher velocity have lower pressure!

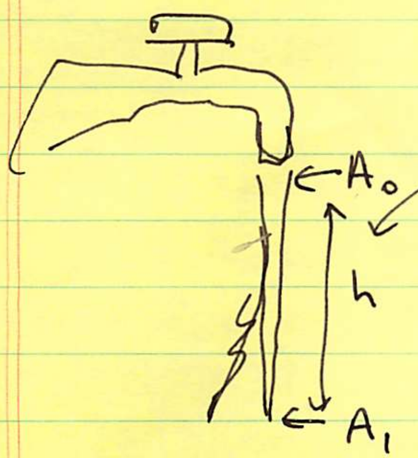


leaving crowded movie theatre

Think about a ~~"Road block"~~ → before that move slowly → "pressure" is great (keep bumping against one another) but after leaving, v is high, pressure small.

[n.b. this flow is compressible & viscous].

Example: flow of water from tap.



As speed of water increases, cross-sectional falls.

Suppose we ~~know~~ measure A_0, A_1 & h, can measure flow rate from tap.

$$A_0 v_0 = A_1 v_1$$
$$v_1^2 = v_0^2 + 2gh$$

$$\Rightarrow \quad A_0^2 v_0^2 = A_1^2 \left[v_0^2 + 2gh \right]$$

$$\Rightarrow \quad v_0^2 \left[A_0^2 - A_1^2 \right] = A_1^2 \, 2gh$$

$$\Rightarrow \quad v_0 = A_1 \left[\frac{2gh}{A_0^2 - A_1^2} \right]^{1/2}$$

$$\Rightarrow \quad R = A_0 v_0 = A_0 A_1 \left[\frac{2gh}{A_0^2 - A_1^2} \right]^{1/2}$$

Amazingly, we ~~kno~~ don't have to measure any velocities to know the flow rate!

More generally: equation of continuity

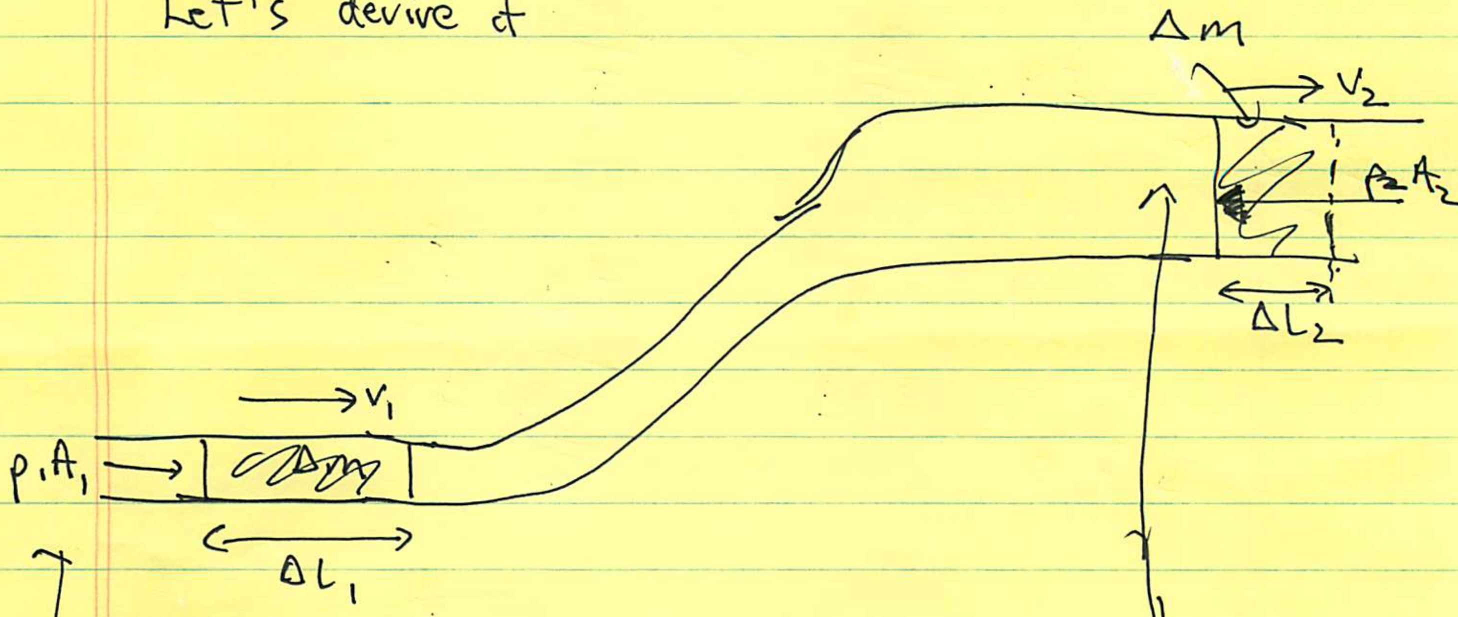
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \vec{x}} (\rho \vec{v}) = 0.$$

Applies to mass, electric charge, etc...
anytime there is a conserved scalar quantity.

Bernoulli's equation

Essentially a statement of work energy theorem. (i.e., not a new law, but can be derived from Newtonian mechanics). Nonetheless, it's <u>extremely important</u>!

Let's derive it



Work energy theorem
→ work done by forces [here, pressure + gravity] = change in kinetic energy.

$$\Delta W = p_1 A_1 \Delta L_1 - p_2 A_2 \Delta L_2 - \Delta m g (y_2 - y_1)$$

$$\rho \Delta V = \Delta m = \rho A \Delta L$$

Always the same, since water is incompressible

$$\Rightarrow \Delta L_1 = \frac{\Delta m}{\rho A_1}, \quad \Delta L_2 = \frac{\Delta m}{\rho A_2}$$

$$\Rightarrow \Delta W = p_1 \frac{\Delta m}{\rho} - p_2 \frac{\Delta m}{\rho} - \Delta m\, g(y_2 - y_1)$$

$$\Delta K = \frac{1}{2} \Delta m \left(v_2^2 - v_1^2\right)$$

$$\Rightarrow \Delta m \left[\frac{p_1}{\rho} - \frac{p_2}{\rho} - g(y_2 - y_1) \right] = \frac{1}{2} \Delta m \left(v_2^2 - v_1^2\right)$$

$$\Rightarrow p_1 + \frac{1}{2}\rho_1 v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho_2 v_2^2 + \rho g y_2$$

$$\Rightarrow \boxed{p + \frac{1}{2}\rho v^2 + \rho g y = \text{const}}$$

Bernoulli's equation for steady, incompressible, non-viscous & irrotational flow.

We used streamline along pipe axis.
If flow is irrotational, ~~const~~ can show that const is same for <u>all</u> streamlines.

Fluid statics can emerge as special case of fluid dynamics.

If fluid is at rest, $v_1 = v_2 = 0$

$$\Rightarrow p_1 + \rho g y_1 = p_2 + \rho g y_2$$

$$\Rightarrow p_2 - p_1 = -\rho g (y_2 - y_1)$$

as we previously derived!

Also consider if tube is horizontal, $y_1 = y_2$

$$\Rightarrow \quad p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

~~dyna~~

Large speed $\Rightarrow$ low pressure
Slow speed $\Rightarrow$ high pressure

$$\cancel{p + \tfrac{1}{2}\rho} \quad \underbrace{p + \rho g y}_{\text{static pressure}} + \underbrace{\frac{1}{2}\rho v^2}_{\text{dynamic pressure}}$$

Essentially a statement of conservation of energy

$$\Delta K + \Delta U = W$$

K.E.   potential energy   work done by pressure force.

If fluid is compressible, it can acquire internal energy (it takes energy to ~~not~~ push molecules closer together, increasing internal potential energy)

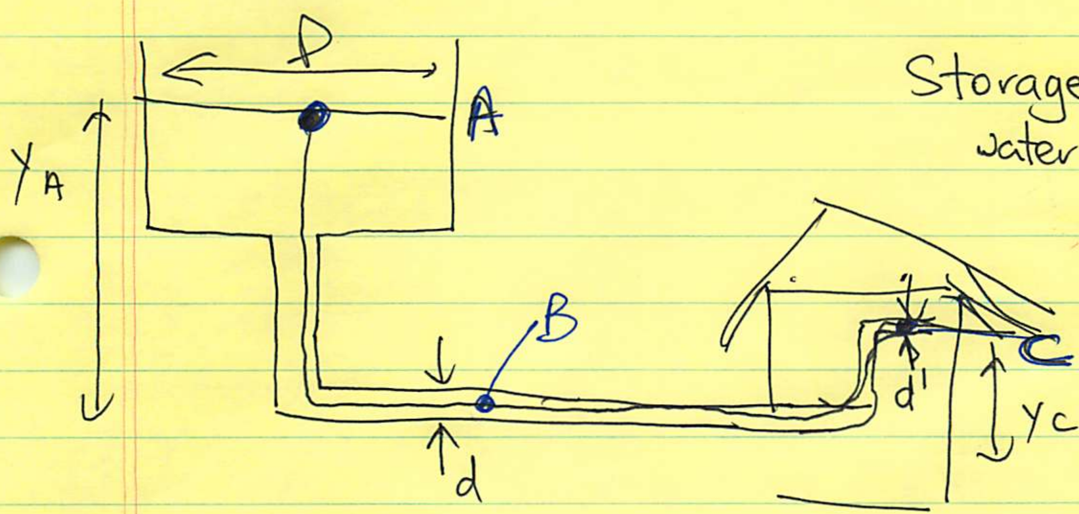
$$\Delta K + \Delta U + \Delta E_{int} = W$$

If flow is viscous, then frictional ~~fric~~ forces ~~are equivalent to~~ also cause increase in internal energy.

~~Bernoulli's Equation~~

So Bernoulli eqn can be easily modified to account for mechanical energy → internal energy.
We're working in incompressible limit where these corrections are negligible.

Example



Storage tower supplies water to house.

$y_A = 30$ m (storage tower height)
$D = 3$ m (diameter of ~~pipe fr.~~ storage tower).
$d = 3$ cm (diameter of pipe fr. storage tower).
$d' = 1$ cm (diameter of pipe at house)
Water flow rate $R \simeq 3$ liter/s.
[N.B. → 1 liter = 10cm × 10cm × 10cm
$= 10^{-3}$ m^{-3}].
$y_C = 10$ m (height of 2nd storey of house)

(a) Suppose water is flowing at this rate. What is the pressure in the horizontal pipe?

Apply Bernoulli's equation

$$p_A + \frac{1}{2}\rho v_A^2 + \rho g y_A = p_B + \frac{1}{2}\rho v_B^2 + \rho g y_B$$

Find v_A, v_B fr. conservation of mass :

$$v_A A_A = v_B A_B = R$$

$$\Rightarrow \quad v_A = R/A_A = \frac{3\times10^{-3}}{\pi(3)^2} \simeq 10^{-4} \text{ ms}^{-1}.$$

$$v_B = R/A_B = \frac{3\times10^{-3}}{\pi(3\times10^{-2})^2} \simeq 1 \text{ ms}^{-1}.$$

Thus, $\rho v_A^2 \ll \rho v_B^2$

we can neglect this.

$$\underbrace{p_A}_{= p_0,} + \overbrace{\cancel{\tfrac{1}{2}\rho v_A^2}}^{\text{neglect}} + \rho g y_A = p_B + \frac{1}{2}\rho v_B^2 + \underbrace{\cancel{\rho g y_B}}_{\to 0}$$

$$\Rightarrow p_B = p_0 + \rho g y_A - \frac{1}{2}\rho v_B^2$$

$$= 10^5 + (10^3)(10)(30) - \frac{1}{2}(10^3)(1)^2$$

$$= 4\times10^5 \text{ Pa}.$$

v. small correction..

(b) What is velocity in narrow pipe at C?

$$p_A + \underbrace{\cancel{\tfrac{1}{2}\rho_A v_A^2}}_{\text{negligible}} + \rho g y_A = p_C + \tfrac{1}{2}\rho_C v_C^2 + \rho g y_C$$

$$\Rightarrow \quad p_C = p_0 - \tfrac{1}{2}\rho_C v_C^2 + \rho g (y_A - y_C)$$

Again, find

$$v_C = \frac{R}{A_C} = \frac{3\times10^{-3}}{\pi\,(10^{-2})^2} = 10 \text{ m s}^{-1}!$$

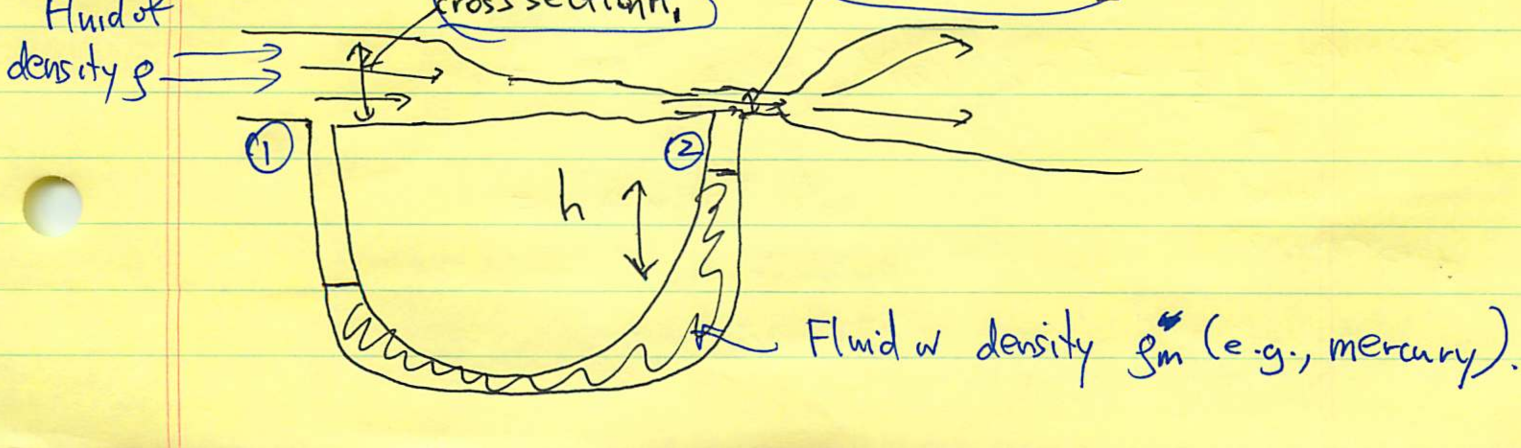
Then $p_C = 10^5 - \tfrac{1}{2}(10^3)(10)^2 + (10^3)(10)(20)$

$$= 2.5\times10^5 \text{ Pa}.$$

Pressure is lower because of lower height drop (static effect) & larger flow velocity through narrower pipe (dynamic effect).

Applications of Bernoulli's Equation & Equation of Continuity

Venturi Meter



What is the speed of ~~vo~~ flow at point 1?

[N.B. → remember that we're considering incompressible flow].

~~It is~~ $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$ (1)

$$\rho A_1 v_1 = \rho A_2 v_2$$

$$\Rightarrow v_2 = v_1 \left(\frac{A_1}{A_2}\right)$$

~~Also, from manometer eqn~~ At manometer $p_1 + \rho g h = p_2 + \rho_m g h$

~~$p_1 - p_2 = \rho_m g h$~~ $\Rightarrow p_1 - p_2 = (\rho_m - \rho) g h$

~~⇒~~ From (1),

$$\frac{1}{2}\rho\left[v_2^2 - v_1^2\right] = (p_1 - p_2)$$

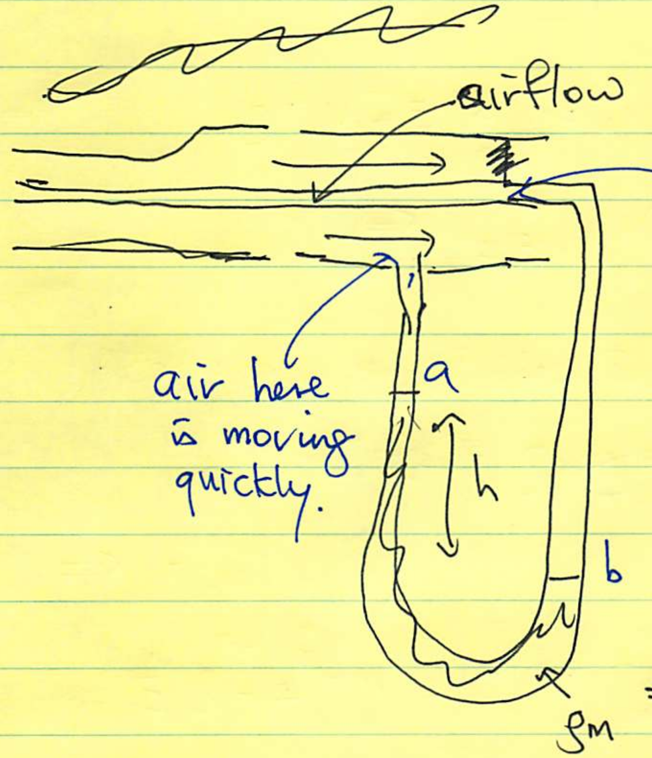
$$\Rightarrow \frac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] = \cancel{\rho_m g h}\ (\rho_m - \rho) g h$$

$$\Rightarrow v_1^2 = \frac{\cancel{2\rho_m g h}\ 2(\rho_m - \rho) g h}{\rho\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}$$

$$\Rightarrow \boxed{v_1 = A_2\left(\frac{2(\rho_m - \rho) g h}{\rho(A_1^2 - A_2^2)}\right)^{1/2}}$$

Pitot Tube → used to measure flow speed of gas.

E.g. airspeed indicator on aircraft wingtips.



$$p_a + \frac{1}{2}\rho v_a^2 = p_b$$

$$p_b - p_a = \rho_m g h$$

$$\Rightarrow \frac{1}{2}\rho v_a^2 = p_b - p_b = \rho_m g h$$

$$\Rightarrow v_a = \sqrt{\frac{2gh\rho_m}{\rho}}$$

Dynamic Lift.

Force that acts on body [airplane wing, hydrofoil, helicopter rotor] via its motion.

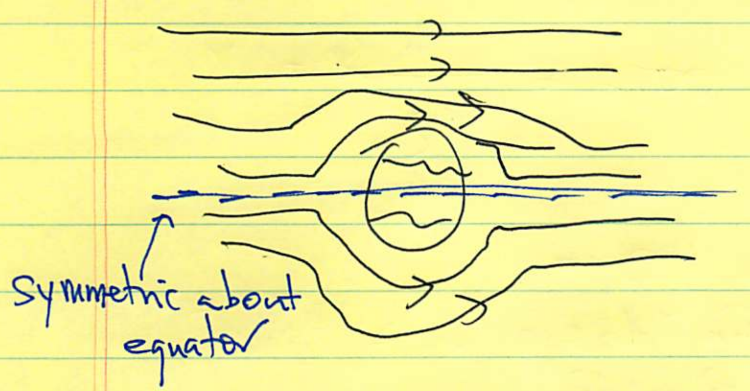
→ distinct fr. static lift due to Archimedes principle (e.g. balloon, iceberg).

Also happens for rotating baseball, tennis ball, golf ball

↳ "curve ball" due to rotation!

Lift is due to viscosity. No viscosity, no lift!

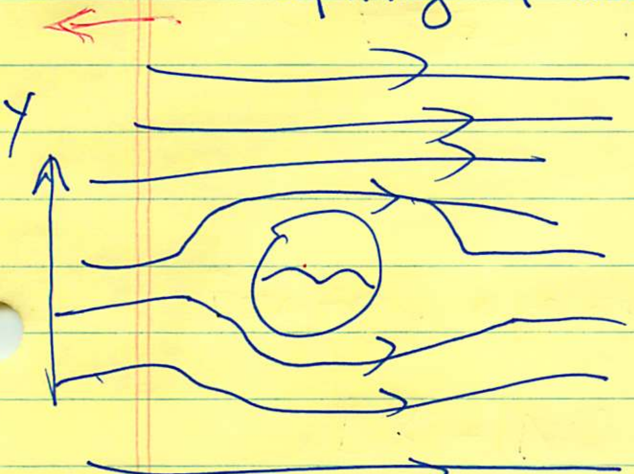
Consider non-rotating ball.
Flow



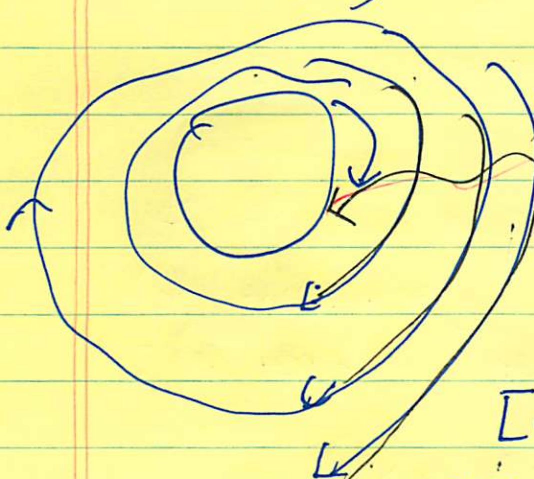
Dynamic Lift.

~~Lift~~ Force on body due to motion [distinct fr. static lift due to buoyant forces].

E.g. curveball in baseball [or tennis, or golf]. Spinning of the ball produces dynamic lift!



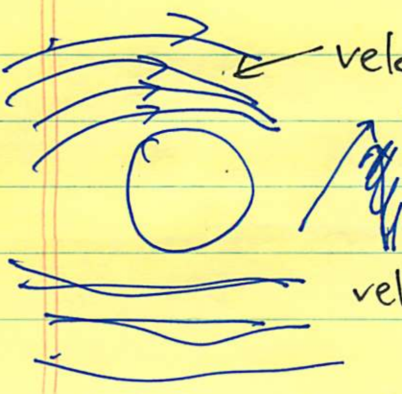
Symmetric wrt equator — no net force in y direction



$v = 0$ at surface of ball [in rest ~~relative to~~ frame of non-rotating ~~ball~~ ball].

Viscosity causes a thin boundary layer to rotate.

[Golf balls are dimpled to increase this circulation. Baseballs also illegally roughened by pitchers for same reason].



velocities add here

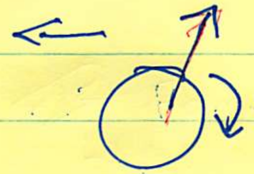
v is higher → pressure is lower

net force

velocities subtract her → pressure is higher

v is larger

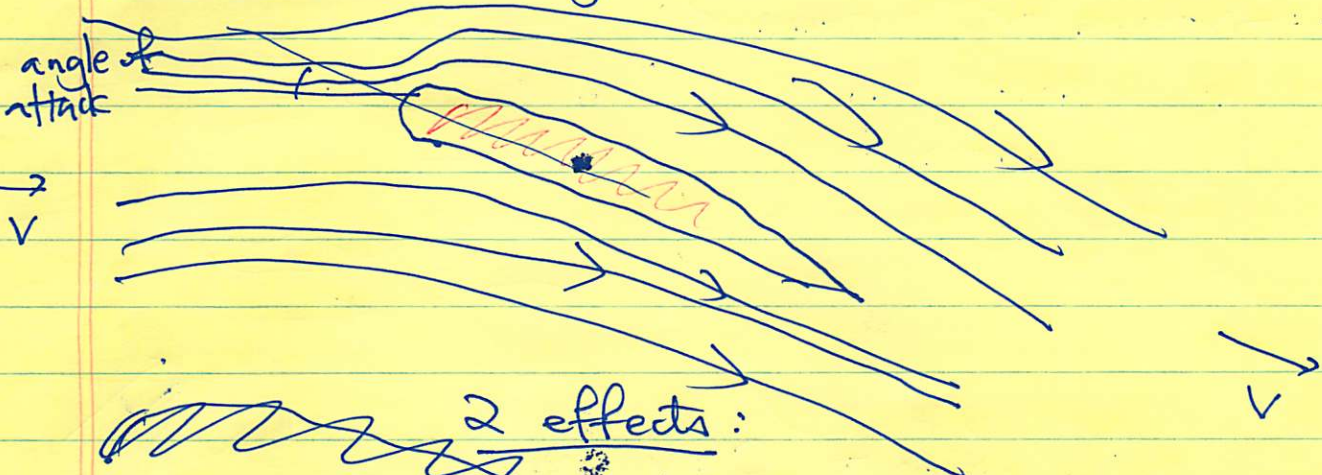
Balls curve in flight for same ~~reaso~~ reason



Looking from top, ball has sidespin – will curve to side

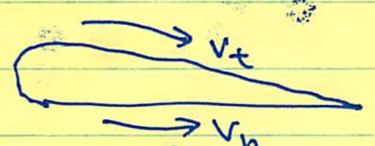
Looking fr. side, ball has backspin – will curve down.

Lift on aircraft wing

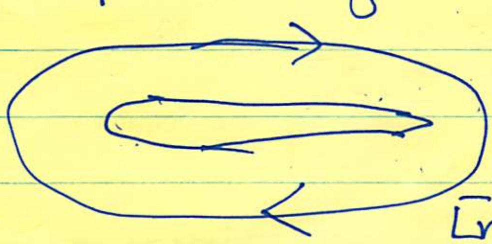


2 effects:

①



Shape of wing: $v_t > v_b$ – circulation develops. Just like for baseball

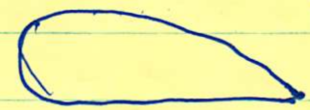


Lower pressure on top → lift force.

[note: the usual explanation that ~~there~~ ~~top & bott~~ is more distance to travel on top, so v is faster, is WRONG.

→ air tunnel experiments show that air on top and bottom doesn't reach end at same time

(in fact, top reaches end much faster than bottom!).
— if this explanation is true, then wing shape needed is crazy!

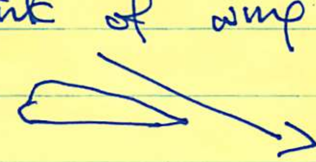


— how do airplanes fly ~~th~~ upside down? (there are also plane w/ flat wings ...)

Bernoulli's theorem does hold, but for different complicated reasons (due to the development of vortices).

② Angle of attack on wing

Air is deflected downward!
So from Newton's 3rd law, there should be an upward force on wing. Think of wing as scooping air downward.



The important surface is the top surface.
(that is why we can put stuff like engines & wheels below the wing, but the top surface must be clear).

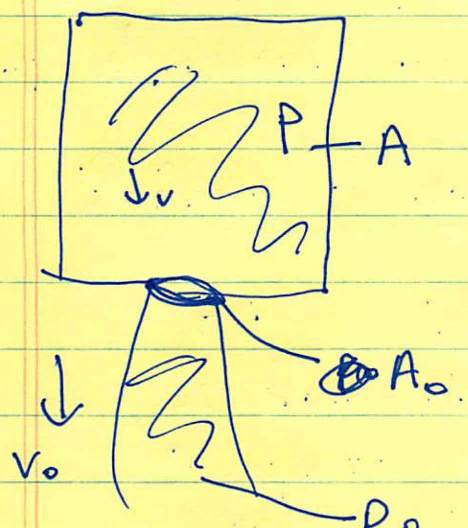
(N.B. if increase angle of attack too much, air cannot ~~flow w~~ follow wings, but breaks off into mini whirlpools → STALL).

Faster speed → encounters more air, more lift.

But this isn't possible at takeoff & landing
Flaps & slots increase surface area → more lift
at lower speed.

Slot → "guides" air around wing so there is no boundary layer separation, & plane doesn't stall.

Thrust on a Rocket



Can ignore variation in pressure w/ height in chamber

$$p - p_0 = \frac{1}{2}\rho\left(v_0^2 - v^2\right)$$

$$\Rightarrow v_0^2 = \frac{2(p-p_0)}{\rho} + v^2$$

Let's treat flow as steady & incompressible for pressure & exhaust speeds that aren't too high (more generally, gas is compressible & at high speeds flow can become turbulent).

$$Av = A_0 v_0 \Rightarrow \frac{v_0}{v} = \frac{A}{A_0} \gg 1$$

so can ignore v

$$\Rightarrow v_0^2 = \frac{2(p-p_0)}{\rho}$$

Thrust on rocket is

$$F = \frac{dp}{dt} = \frac{d(Mv)}{dt} = v_0 \frac{dM}{dt}$$

$$dM = \rho A_0 v_0 dt \Rightarrow \frac{dM}{dt} = \rho A_0 v_0$$

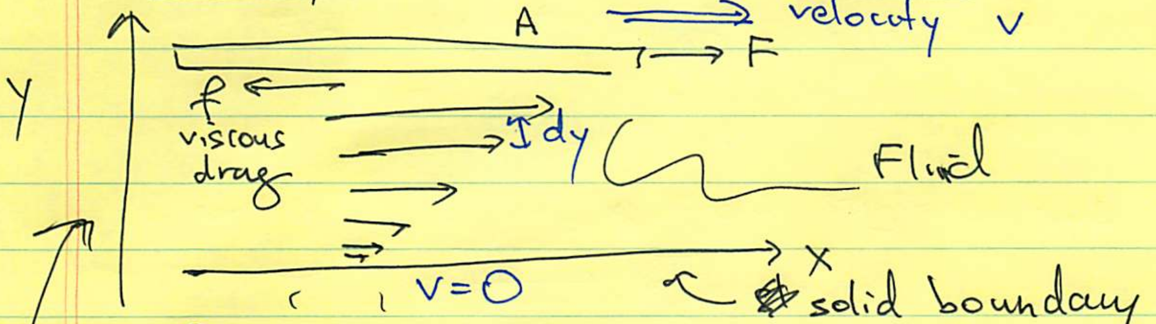
$$\Rightarrow F = v_0 \frac{dM}{dt} = \rho A_0 v_0^2 = \rho A_0 \frac{2(p-p_0)}{\rho}$$

$$= 2 A_0 (p - p_0)$$

Note factor of 2!

(V1)

Viscosity, Turbulence, & Fluid Flow



~~Similar~~ Viscosity is similar to friction in motion of solid bodies. Need to apply force to keep things moving.

Consider situation above. Force $\vec{F}$ opposes viscous drag f so that velocity of plate is const.

Divide fluid up into layers.

Laminar flow: velocity varies smoothly layer-by-layer.

Define shearing stress $= F/A$ — area over which force acts.

Fluid responds by developing a velocity gradient:

Shearing strain $\dfrac{dv}{dy}$

[N.B. solid responds to stress by deforming in shape
fluid responds by flowing].

Coefficient of vicosity

$$\eta = \frac{\text{Shearing stress}}{\text{Shearing strain}}$$

(how much Force / Area you need to apply to produce a given velocity gradient)

(V2)

$$\eta = \frac{F/A}{\frac{dv}{dy}}$$

In this case, if we assume $\frac{dv}{dy} = \text{const} = \frac{V}{D}$

$$\Rightarrow \eta = \frac{F/A}{V/D} = \frac{FD}{AV}$$

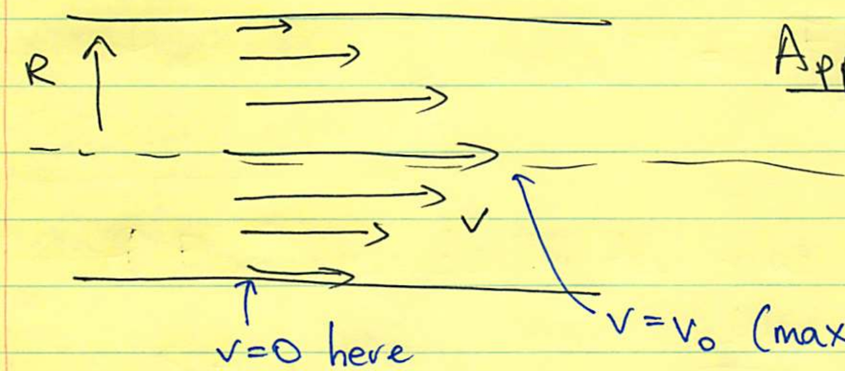
$$[\eta] = \frac{N\,m}{m^2\,m\,s^{-1}} = N\,m^{-2}\,s$$

~~Air η =~~

Air (20°) $\eta = 1.8 \times 10^{-5}\ N m^{-2} s$

Water (20°) $\eta = 10^{-3}\ N\,m^{-2}\,s$.

Glycerine (20°) $\eta = 1.5\ N\ m^{2}\ s$.



Application Flow of fluid through pipes.

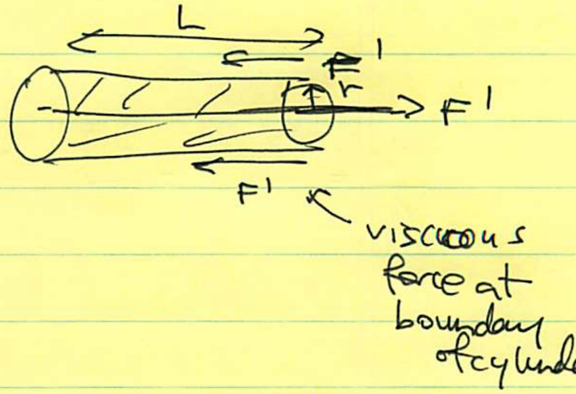
Assume laminar flow.

Ask class : is flow rotational?

(yes, if put paddlewheel anywhere except central streamline, it will rotate due to variations in velocity).

In this case, variations in velocity are not linear!

Let's solve this problem (P41 in HRK).

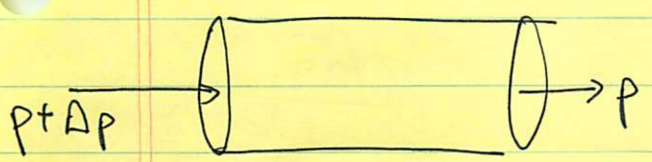


$$\eta = \frac{F/A}{\frac{dv}{dr}}$$

$A =$ area of ~~cylinder~~ cylinder $= 2\pi r L$.

$$\Rightarrow F_{viscous} = -A \frac{dv}{dr} \eta$$

$$= -2\pi r L \frac{dv}{dr} \eta.$$



Force F' pushing cylinder of fluid through pipe:

$$F' = (p+\Delta p)A - (p)A = \Delta p\, A = \Delta p\, \pi r^2.$$

In equilibrium, $F' = F$

$$\Rightarrow \Delta p\, \pi r^2 = -2\pi r L \frac{dv}{dr} \eta$$

$$\Rightarrow \frac{dv}{dr} = -\frac{\Delta p}{2L\eta} r$$

$$\Rightarrow v = v_0 - \frac{\Delta p}{4L\eta} r^2$$

const of integration

~~$v(r=R) = v_0$~~

$$v(r=R) = 0 \Rightarrow v_0 - \frac{\Delta p}{4L\eta} R^2 = 0 \Rightarrow v_0 = \frac{\Delta p R^2}{4L\eta}$$

$$\Rightarrow \quad v = v_0\left(1-\frac{r^2}{R^2}\right) \quad \text{where} \quad v_0 = \frac{\Delta p R^2}{4L\eta} \ \cancel{\text{and}}$$

and Δp is the pressure drop across length L.

Total mass flux is

$$\dot{M} = \int \rho v(r)\, dA$$

$$= \rho \int_0^R v_0\left(1-\frac{r^2}{R^2}\right) 2\pi r\, dr$$

$$= 2\pi\rho v_0 \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R$$

$$\cancel{= \frac{4\pi}{3}\rho v_0}$$

$$= \frac{\pi}{2}\rho v_0 R^2 = \frac{\rho \pi R^4 \Delta p}{8\eta L}$$

Poiseuille's law. 2 applications:

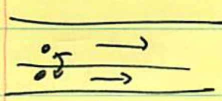
1) Given η, can determine Δp needed to pump a fluid through distance L.
2) By applying a known Δp, measuring $\dot{M}$ allows us to determine η.

Ask class: does η increase or decrease with temperature?

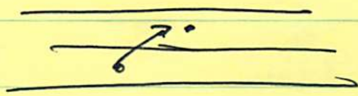
Ans: it ~~increases, because~~ decreases. ~~Viscosity depends~~

~~on interna~~

It depends. Viscosity depends on ability of one layer to couple to another.



liquids



gases.

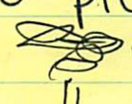
~~In liquids~~ In particular, one layer has to be able to transfer momentum to another layer.

In liquids, this happens via intermolecular forces. There are weaker at higher T, due to increased kinetic energy of molecules → viscosity is lower at higher T.

In gases, coupling happens via molecules fr. one layer migrating to another → collide with molecules there, transfer momentum. This is greater at higher T → viscosity is higher at higher T.

Turbulence

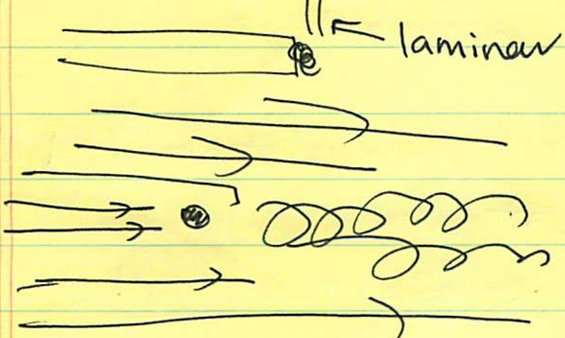
↳ Show pictures!



← turbulent.

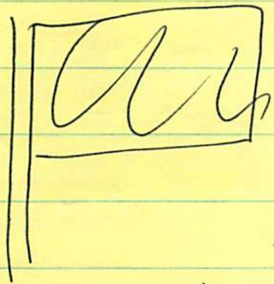
← laminar

Cigarette smoke — smooth column breaks up into irregular eddies.



Fluid flowing past obstacle breaks up into vortices & eddies.

V6



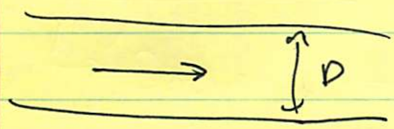
Flag flapping in wind. If flow was laminar, flag would lie on fixed position along streamlines. Instead, flag breaks flow into irregular pattern, and flaps along with it.

<u>TURBULENT</u> fluid flow! Other examples
- wake of ships / aircraft.
- whistling, wind instruments.

At low velocities, flow is laminar
high velocities, flow becomes turbulent.

[Analogy, → push block on ground at low v → friction balances applied force.
As push faster & faster, block eventually tips over].

What is the critical velocity v_c at which flow becomes turbulent? Consider flow in pipe.



Expect v_c to depend on η, D, ρ

$$v_c \propto \eta^a \rho^b D^c$$

Use dimensional analysis:

$$\Rightarrow [v_c] = [\eta^a][\rho^b][D^c]$$

What are the dimensions of viscosity η?

$[\eta] = N \cdot s\, m^{-2}$

$\Rightarrow kg\, m\, s^{-2}\, s\, m^{-2} = kg\, m^{-1} s^{-1}$

$m\, s^{-1} = (kg\, m^{-1} s^{-1})^a (kg\, m^{-3})^b (m)^c$

for kg, $a + b = 0$
for m $1 = -a - 3b + c$
for s $-1 = -a$

$\Rightarrow a = 1$
$b = -1$
$1 = -1 + 3 + c \Rightarrow c = -1$

$\Rightarrow v_c \propto \frac{\eta}{\rho D} \Rightarrow v_c = R \frac{\eta}{\rho D}$

this dimensionless coefficient is known as the Reynolds number

$\boxed{R = \frac{\rho D v}{\eta}}$ → Reynolds # for any flow velocity.

For cylindrical pipes, R=2000 marks transition to turbulent flow.

Note that critical flow speed increases with viscosity → laminar flow becomes more likely with high ~~velocity~~ viscosity.

V8

Thus the critical speed is for $D = 2\text{cm}$ (typical pipe width)

$$v_c = R\frac{\eta}{\rho D} = 2000 \frac{1\times 10^{-3}\ \text{N}\cdot\text{s}\,\text{m}^{-2}}{10^3\ \text{kg}\,\text{m}^{-3}\,(0.02\text{m})}$$

$$= 0.1\ \text{m}\,\text{s}^{-1} = 10\ \text{cm}\,\text{s}^{-1}.$$

Typical flow speeds are $v \simeq 10\ \text{m}\,\text{s}^{-1}$, water flow fr. household plumbing is turbulent!