QUICK OVERVIEW OF ELECTRODYNAMICS IN VACUUM

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Abstract

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I. INTERNAL CONSISTENCY

In SI units, the equations of electrodynamics in vacuum are

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \text{(Gauss' Law)} \tag{1.1}$$

$$\nabla\cdot\vec{B} = 0 \qquad \text{(No magnetic monopoles)} \tag{1.2}$$

$$\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{J} \qquad \text{(Ampere–Maxwell Law)} \tag{1.3}$$

$$\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \qquad \text{(Faraday's Law)} \tag{1.4}$$

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{J} = 0 \qquad \text{(Conservation of Charge)} \tag{1.5}$$

$$\vec{F} = Q\big(\vec{E} + \vec{v}\times\vec{B}\,\big) \qquad \text{(Lorentz Force Law)} \tag{1.6}$$

If we take the time derivative of eq. (1.1) and the divergence of eq. (1.3), we can combine them to get $\partial\rho/\partial t + (\varepsilon_0\mu_0c^2)\nabla\cdot\vec{J}$; consistency with eq. (1.5) then requires

$$\varepsilon_0\mu_0c^2 = 1\ . \tag{1.7}$$

If we set $\rho = 0$ and $\vec{J} = 0$, we can combine eqs. (1.1–1.4) to get the wave equation for each component of the electric and magnetic field,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\big(\vec{E}\text{ or }\vec{B}\,\big) = 0\ . \tag{1.8}$$

This implies that electromagnetic waves all propagate at the same speed c, which we identify as *the speed of light in vacuum*. The meter is defined as the distance that light travels in a certain time; this gives the speed of light an exact value of

$$c = 299\,792\,458\ \text{m/s}\ . \tag{1.9}$$

Similarly, the coulomb is defined to be the amount of charge that gives μ_0 (the *vacuum permeability*) an exact value of

$$\mu_0 = 4\pi\times10^{-7}\,\text{H/m}\ . \tag{1.10}$$

Here H is the henry (a unit of inductance), given by $\text{H} = \text{kg}\,\text{m}^2/\text{C}^2$, where C is the coulomb. From eq. (1.7), we then get the value of ε_0 (the *vacuum permittivity*),

$$\varepsilon_0 = 8.854\,187\,817\,620\ldots\times10^{-12}\,\text{F/m}\ . \tag{1.11}$$

Here F is the farad (a unit of capacitance), given by $\text{F} = \text{s}^2\,\text{C}^2/\text{kg}\,\text{m}^2$.

The Lorentz Force Law gives us a working definition of electric and magnetic fields (by giving the force on a test charge). The relative coefficient of $\nabla\times\vec{E}$ and $\partial\vec{B}/\partial t$ in eq. (1.4) then follows from requiring Galilean invariance of the Lorentz force at low velocities (which was demonstrated by Faraday in his experiments).

II. OTHER UNIT SYSTEMS

Another system of units in common use is the gaussian system, which sets $\varepsilon_0 = 1/4\pi$, $\mu_0 = 4\pi/c^2$, and makes the replacement $\vec{B} \to \vec{B}/c$.

When studying quantum electrodynamics, Heaviside–Lorentz units are often used; in this system, we set $\varepsilon_0 = \mu_0 = c = 1$ (and also $\hbar = 1$, where $\hbar$ is the reduced Planck's constant).

III. SCALAR AND VECTOR POTENTIALS

According to the *Helmholtz theorem* of vector calculus (sse section 1.9 of *Zangwill*), a vector field with zero divergence is the curl of another vector field; since $\nabla \cdot \vec{B} = 0$, we have

$$\vec{B} = \nabla \times \vec{A} \,, \tag{3.1}$$

where $\vec{A}$ is the *vector potential*. Plugging eq. (3.1) into Faraday's Law, we get

$$\nabla \times \left(\vec{E} + \frac{\partial}{\partial t} \vec{A} \right) = 0 \,. \tag{3.2}$$

The Helmholtz theorem also tells us that a vector field with zero curl is the gradient of a scalar field; thus we have $\vec{E} + (\partial/\partial t)\vec{A} = -\nabla\varphi$, or equivalently

$$\vec{E} = -\frac{\partial}{\partial t} \vec{A} - \nabla\varphi \,, \tag{3.3}$$

where φ is the *scalar potential*.

If we write $\vec{E}$ and $\vec{B}$ using eqs. (3.1) and (3.3), then eq. (1.2) (no monopoles) and eq. (1.4) (Faraday's Law) are automatically satisfied.

IV. GAUGE INVARIANCE

The scalar and vector potentials are not uniquely determined by the electric and magnetic fields. If φ and $\vec{A}$ are potentials that result in a particular $\vec{E}$ and $\vec{B}$, then so do the *gauge transformed* potentials

$$\varphi' = \varphi - \frac{\partial}{\partial t} \chi \,, \tag{4.1}$$

$$\vec{A}' = \vec{A} + \nabla\chi \,, \tag{4.2}$$

where χ is an arbitrary scalar field. The $\vec{E}$ and $\vec{B}$ fields are *gauge invariant*.

Gauge invariance is now believed to be a deep property of nature. The Standard Model of elementary particles, which incorporates the strong and weak nuclear forces as well as electromagnetism, uses gauge invariance as a fundamental principle. General relativity incorporates *general coordinate invariance*, which is a kind of gauge invariance.

V. LAGRANGIAN FORMULATION OF ELECTRODYNAMICS

Electrodynamics is an example of a classical mechanical system, and as such it has a lagrangian formulation.

From here on, we will use Heaviside–Lorentz units, and set

$$\varepsilon_0 = \mu_0 = c = 1 \; . \tag{5.1}$$

These constants can always be restored in any particular equation by dimensional analysis.

Recall that a classical mechanical system is based on some *generalized coordinates* $q_I(t)$, where I is an index. (These generalized coordinates may or may not represent the position of something.) The *action* of the system is then

$$S = \int dt \, L(q_I(t), \dot{q}_I(t), t) \; , \tag{5.2}$$

where $L(q_I, \dot{q}_I, t)$ is the *lagrangian*; it is a function of the coordinates $q_I(t)$ at a particular time t, the first time derivative of these coordinates $\dot{q}_I(t)$ at this time, and (possibly) the time t itself. The *equations of motion* follow from requiring that an arbitrary change in the coordinates $\delta q_I(t)$ leave the action unchanged:

$$\delta S = 0 \; . \tag{5.3}$$

We begin by considering electrodynamics with prescribed charges and currents. That is, we take ρ and $\vec{J}$ to be fixed functions consistent with eq. (1.5).

The appropriate generalized coordinates are then the potentials $\varphi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$; here $\vec{x}$ should be thought of as a continuous part of the index I. The lagrangian is given as a volume integral of a *lagrangian density* $\mathcal{L}$,

$$L = \int d^3x \, \mathcal{L} \; . \tag{5.4}$$

Thus the action can be written as a four-dimensional volume integral,

$$S = \int d^4x \, \mathcal{L} \; , \tag{5.5}$$

where $d^4x = dt \, d^3x$. We note in passing that d^4x is Lorentz invariant: under a Lorentz boost in (say) the x direction, the time dilation of dt is canceled by the length contraction of dx.

The appropriate lagrangian density for electrodynamics turns out to be

$$\mathcal{L} = \tfrac{1}{2}\big(\vec{E}^2 - \vec{B}^2\big) + \vec{J}\cdot\vec{A} - \rho\varphi \; , \tag{5.6}$$

where the fields $\vec{E}$ and $\vec{B}$ should be expressed in terms of the potentials φ and $\vec{A}$ via eqs. (3.1) and (3.3). If we vary the potentials, the variation of the action is

$$\delta S = \int d^4x \, \Big[\vec{E}\cdot\delta\vec{E} - \vec{B}\cdot\delta\vec{B} + \vec{J}\cdot\delta\vec{A} - \rho\delta\varphi\Big] \tag{5.7}$$

$$= \int d^4x \, \Big[\vec{E}\cdot\big(-\tfrac{\partial}{\partial t}\delta\vec{A} - \nabla\delta\varphi\big) - \vec{B}\cdot\big(\nabla\times\delta\vec{A}\big) + \vec{J}\cdot\delta\vec{A} - \rho\delta\varphi\Big] \tag{5.8}$$

$$= \int d^4x \, \Big[\Big(\tfrac{\partial}{\partial t}\vec{E} - \nabla\times\vec{B} + \vec{J}\Big)\cdot\delta\vec{A} + \Big(\nabla\cdot\vec{E} - \rho\Big)\delta\varphi\Big] \; . \tag{5.9}$$

Eq. (5.8) follows from the expressions for the fields in terms of the potentials, and eq. (5.9) follows from some integrations by parts; we assume $\delta\varphi$ and $\delta\vec{A}$ vanish on the boundaries of spacetime at infinity, so there are no surface terms. Now we see that requiring $\delta S = 0$ for generic $\delta\vec{A}$ and $\delta\varphi$ yields the Ampere–Maxwell Law and Gauss' Law, while Faraday's Law and the absence of monopoles follow automatically from expressing the fields in terms of the potentials. Thus we have recovered the four Maxwell equations.

Now suppose that the charge and current densities are due to a single point particle with charge Q that moves along a path $\vec{q}(t)$. We then have

$$\rho(\vec{x}, t) = Q\,\delta(\vec{x} - \vec{q}(t)) \;, \tag{5.10}$$

$$\vec{J}(\vec{x}, t) = Q\,\dot{\vec{q}}(t)\,\delta(\vec{x} - \vec{q}(t)) \;, \tag{5.11}$$

where $\delta(\vec{x}) \equiv \delta(x)\delta(y)\delta(z)$ is the three-dimensional Dirac delta function. We will derive the Lorentz Force Law for this particle from the action, treating $\vec{q}(t)$ as another generalized coordinate.

To the action we have already, given by eqs. (5.5) and (5.6), we must add a kinetic term for the particle, which will take the form

$$S_{\rm kin} = \int dt\, L_{\rm kin}(\dot{\vec{q}}) \;. \tag{5.12}$$

For a nonrelativistic particle, we would take $L_{\rm kin} = \frac{1}{2}m\dot{\vec{q}}^2$. For a relativistic particle, the appropriate expression is $L_{\rm kin} = -mc^2/\gamma$, where $\gamma \equiv (1 - \dot{\vec{q}}^2/c^2)^{-1/2}$; see eq. (24.27) of *Zangwill*. The key point is that, whatever choice is made for $L_{\rm kin}$, the particle's kinetic momentum is given by

$$p_i = \frac{\partial L_{\rm kin}}{\partial \dot{q}_i} \;. \tag{5.13}$$

This yields $\vec{p} = m\dot{\vec{q}}$ in the nonrelativistic case and $\vec{p} = \gamma m\dot{\vec{q}}$ in the relativistic case.

The $\vec{q}$ dependent terms in the total action are then

$$\begin{aligned} S &= \int dt\, L_{\rm kin}(\dot{\vec{q}}) + \int d^4x \left(\vec{J}\cdot\vec{A} - \rho\varphi\right) \\ &= \int dt \left[L_{\rm kin}(\dot{\vec{q}}) + Q\dot{\vec{q}}\cdot\vec{A}(\vec{q}, t) - Q\varphi(\vec{q}, t)\right] \;, \end{aligned} \tag{5.14}$$

where the second line follows from plugging in eqs. (5.10) and (5.11). Varying $\vec{q}$ and writing out the vector indices (repeated indices are implicitly summed), we get

$$\begin{aligned} \delta S &= \int dt \left[\frac{\partial L_{\rm kin}}{\partial \dot{q}_i}\delta\dot{q}_i + QA_i\delta\dot{q}_i + Q\dot{q}_j(\nabla_i A_j)\delta q_i - Q(\nabla_i\varphi)\delta q_i\right] \\ &= \int dt \left[\big(p_i + QA_i\big)\delta\dot{q}_i + Q\big(\dot{q}_j\nabla_i A_j - \nabla_i\varphi\big)\delta q_i\right] \;. \end{aligned} \tag{5.15}$$

Note that we have chosen the dummy indices in eq. (5.15) so that δq always has index i. Next we integrate the time derivative on $\delta\dot{q}$ by parts, and use the convective derivative

$$\frac{d}{dt}A_i(\vec{q}(t), t) = \frac{\partial}{\partial t}A_i + (\nabla_j A_i)\dot{q}_j \;, \tag{5.16}$$

which follows from the chain rule. The result is

$$\begin{aligned}\delta S &= \int dt \left[-\dot{p}_i + Q\big(-\tfrac{\partial}{\partial t}A_i - \nabla_i\varphi\big) + Q\dot{q}_j\big(\nabla_i A_j - \nabla_j A_i\big)\right]\delta q_i \\ &= \int dt \left[-\dot{p}_i + QE_i + Q\dot{q}_j\varepsilon_{ijk}B_k\right]\delta q_i \ . \end{aligned} \tag{5.17}$$

Requiring the coefficient of δq to vanish, we get

$$\frac{d}{dt}\,\vec{p} = Q\big(\vec{E} + \dot{\vec{q}}\times\vec{B}\,\big) \ . \tag{5.18}$$

This is the Lorentz Force Law.

VI. HAMILTONIAN FORMULATION

In general, we define the *conjugate momentum* π_I to a generalized coordinate q_I to be

$$\pi_I = \frac{\partial L}{\partial \dot{q}_I} \ . \tag{6.1}$$

The hamiltonian is then

$$H = \textstyle\sum_I \pi_I \dot{q}_I - L \ , \tag{6.2}$$

and it equals the total energy. We would like to apply this formalism to the lagrangian of eqs. (5.4) and (5.6).

Gauge symmetry presents us with an immediate problem: the potentials are redundant, since the fields are left unchanged by a gauge transformation. One approach to this issue is to remove the redundancy by imposing an extra condition on the potentials, known as a *gauge condition*, such that there is a unique set of potentials that (1) correspond to a given set of fields, and (2) satisfy the gauge condition. Choosing a gauge condition is also known as *fixing a gauge*. Two popular choices are

$$\nabla\cdot\vec{A} = 0 \qquad \text{(Coulomb gauge)} \tag{6.3}$$

$$\frac{\partial}{\partial t}\varphi + \nabla\cdot\vec{A} = 0 \qquad \text{(Lorenz gauge)} \tag{6.4}$$

In this section we will choose Coulomb gauge. (Lorenz gauge is useful for analysis of propagating electromagnetic waves.)

According to the Helmholtz theorem, any vector field can be decomposed into the sum of a *transverse field*, which has zero divergence, and a *longitudinal field*, which has zero curl. For example, we can write $\vec{A} = \vec{A}_\perp + \vec{A}_\parallel$. In Coulomb gauge, we set $\vec{A}_\parallel = 0$, and then we have $\vec{E}_\perp = -\frac{\partial}{\partial t}\vec{A}_\perp$ and $\vec{E}_\parallel = -\nabla\varphi$. The lagrangian is then

$$L = \int d^3x \Big[\tfrac{1}{2}\big(\tfrac{\partial}{\partial t}\vec{A}_\perp\big)^2 + \tfrac{1}{2}(\nabla\varphi)^2 - \tfrac{1}{2}\vec{B}^2 + \vec{J}\cdot\vec{A}_\perp - \rho\varphi\Big] \ . \tag{6.5}$$

To get this form, we used the fact that the volume integral of the dot product of a transverse field with a longitudinal field is zero, and hence we were able to drop the $\vec{E}_\perp\cdot\vec{E}_\parallel$ term.

From eq. (6.5) we find that the conjugate momentum to $A_{\perp i}$ is

$$\Pi_i = \frac{\partial L}{\partial \dot{A}_{\perp i}} = \dot{A}_{\perp i} = -E_{\perp i} \ , \tag{6.6}$$

and the hamiltonian is

$$\begin{aligned} H &= \int d^3x \Big[\Pi_i \dot{A}_{\perp i} - L \Big] \\ &= \int d^3x \Big[\tfrac{1}{2} \big(\vec{E}_\perp^{\,2} + \vec{B}^2 \big) - \vec{J} \cdot \vec{A}_\perp - \tfrac{1}{2} (\nabla\varphi)^2 + \rho\varphi \Big] \ . \end{aligned} \tag{6.7}$$

Note that time derivatives of φ do not appear in L. Hence, varying φ yields a time-independent equation,

$$-\nabla^2 \varphi = \rho \ . \tag{6.8}$$

This is *Poisson's equation*. The solution is

$$\varphi(\vec{x},t) = \int d^3y \, \frac{\rho(\vec{y},t)}{4\pi |\vec{x} - \vec{y}|} \ . \tag{6.9}$$

If we plug this solution into the hamiltonian, we get

$$H = \int d^3x \Big[\tfrac{1}{2} \big(\vec{E}_\perp^{\,2} + \vec{B}^2 \big) - \vec{J} \cdot \vec{A}_\perp \Big] + H_{\rm coul} \ , \tag{6.10}$$

where

$$H_{\rm coul} = \frac{1}{2} \int d^3x \, d^3y \, \frac{\rho(\vec{x},t) \rho(\vec{y},t)}{4\pi |\vec{x} - \vec{y}|} \tag{6.11}$$

is the instantaneous Coulomb energy in the charge distribution ρ. Equivalent forms include

$$H_{\rm coul} = \tfrac{1}{2} \textstyle\int d^3x \, \rho\varphi = \tfrac{1}{2} \int d^3x \, (-\nabla^2\varphi)\varphi = \tfrac{1}{2} \int d^3x \, (\nabla\varphi)^2 = \tfrac{1}{2} \int d^3x \, \vec{E}_\parallel^{\,2} \ , \tag{6.12}$$

where φ is given by eq. (6.9), and $\vec{E}_\parallel = -\nabla\varphi$.

VII. LORENTZ INVARIANCE

We begin by putting the time and space coordinates together into a *four-vector* x^μ,

$$x^\mu = (t, \vec{x}) \ , \tag{7.1}$$

with $\mu = i = 1,2,3$ corresponding to the usual three space coordinates, and $\mu = 0$ to the time coordinate: $x^0 = t$. We also introduce the notation

$$x_\mu = (-t, \vec{x}) \ . \tag{7.2}$$

The *interval* between the coordinate origin $(0, \vec{0}\,)$ and x^μ is then defined to be

$$x^\mu x_\mu = \vec{x}^{\,2} - t^2 \ , \tag{7.3}$$

where the repeated index μ is implicitly summed. We will have a rule that an implicitly summed index pair must always have one index up and one index down. See Appendix D of *Zangwill* for more details of this notation.

A *Lorentz transformation* is a linear, homogeneous change of coordinates that preserves the value of the interval; we say that the interval is *Lorentz invariant*. The most general linear homogeneous change of coordinates is

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \ , \tag{7.4}$$

where $\Lambda^\mu{}_\nu$ is a 4×4 constant matrix, and the repeated index ν is implicitly summed. We require $\tilde{x}^\mu \tilde{x}_\mu = x^\mu x_\mu$, which implies a condition on $\Lambda^\mu{}_\nu$. One particular $\Lambda^\mu{}_\nu$ that satisfies this condition is a *boost* by a velocity $\vec{v}$ in the x direction; the nonzero components for this particular $\Lambda^\mu{}_\nu$ are $\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma$, $\Lambda^0{}_1 = \Lambda^1{}_0 = -\gamma\beta$, and $\Lambda^2{}_2 = \Lambda^3{}_3 = 1$, with $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$.

Next we group the potentials into a four-vector,

$$A^\mu = (\varphi, \vec{A}) \ , \tag{7.5}$$

and also the charge and current densities,

$$J^\mu = (\rho, \vec{J}) \ . \tag{7.6}$$

We define a four-vector derivative

$$\partial^\mu = (-\tfrac{\partial}{\partial t}, \nabla) \ . \tag{7.7}$$

The minus sign is needed for consistency; the down-index version is

$$\partial_\mu = (\tfrac{\partial}{\partial t}, \nabla) \ . \tag{7.8}$$

In this notation, a gauge transformation, eqs. (4.1) and (4.2), is written as

$$A'^\mu = A^\mu + \partial^\mu \chi \ . \tag{7.9}$$

The continuity equation becomes

$$\partial_\mu J^\mu = 0 \ . \tag{7.10}$$

It is a general rule that an object with all indices *contracted* (that is, all indices appear in implicitly summed pairs with one index up and one index down) is Lorentz invariant. Two different observers, one using coordinates x^μ and one using coordinates $\tilde{x}^\mu$, related by eq. (7.4), will agree on the value of such an object.

Next we define the *field strength tensor*,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{7.11}$$

Note that it is antisymmetric: $F^{\nu\mu} = -F^{\mu\nu}$. If we work out the components of $F^{\mu\nu}$, we get

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -\tfrac{\partial}{\partial t} A^i - \nabla^i \varphi = E^i \ , \tag{7.12}$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = \nabla^i A^j - \nabla^j A^i = \varepsilon^{ijk} B^k \ , \tag{7.13}$$

with $F^{i0} = -F^{0i} = -E^i$ and $F^{00} = 0$. Gauss' Law and the Ampere–Maxwell Law can then be written as

$$\partial_\nu F^{\mu\nu} = J^\mu \ . \tag{7.14}$$

The lagrangian density for electrodynamics, eq. (5.6), can now be written as

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \ , \tag{7.15}$$

which is manifestly Lorentz invariant. We have already seen that d^4x is Lorentz invariant: in the present notation, $d^4\tilde{x} = |\det \Lambda| \, d^4x$, where $\det \Lambda$ is a jacobian determinant; but it turns out that $\det \Lambda = \pm 1$ follows from the invariance of the interval. Hence the action $S = \int d^4x \, \mathcal{L}$ is also Lorentz invariant. It follows that the equations of motion derived from S will take the same form in any two coordinate systems that are related by a Lorentz transformation; these equations are then said to be *Lorentz covariant*.

VIII. SOME USEFUL NUMBERS

The fine structure constant is

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137.036} \ . \tag{8.1}$$

where

$$e = 1.602 \times 10^{-19} \, \mathrm{C} \tag{8.2}$$

is the magnitude of the electron charge, and

$$\hbar c = 1973 \, \mathrm{eV \, \AA} \ . \tag{8.3}$$

Here the electron volt is 1.602×10^{-19} Joule and the angstrom is 10^{-10} m.

The rest energies of the electron, proton, and neutron are

$$m_e c^2 = 0.511 \, \mathrm{MeV} \ , \tag{8.4}$$

$$m_p c^2 = 938.3 \, \mathrm{MeV} \ , \tag{8.5}$$

$$m_n c^2 = 939.6 \, \mathrm{MeV} \ . \tag{8.6}$$

The binding energy of hydrogen (in the limit of infinite proton mass) is the *Rydberg*,

$$\mathrm{Ry} \equiv \frac{e^4 m_e}{32\pi^2 \varepsilon_0^2 \hbar^2} = \tfrac{1}{2} \alpha^2 m_e c^2 = 13.6 \, \mathrm{eV} \ . \tag{8.7}$$

This is also the expectation value of the kinetic energy of the electron in the ground state of hydrogen; this shows that the root-mean-square expected speed of the electron in the ground state is $\langle v^2 \rangle^{1/2} = \alpha c$. Thus the fine structure constant also controls the size of relativistic corrections in atoms, and its smallness allows the nonrelativistic Schrodinger equation to be a good approximation for computing the energy levels of hydrogen.

The *reduced Compton wavelength* of the electron is

$$\lambda\!\!\!^{-}_{\rm C} = \frac{\lambda_{\rm C}}{2\pi} \equiv \frac{\hbar c}{m_e c^2} = 3.86 \times 10^{-3}\,\text{Å} \ . \tag{8.8}$$

The size of a hydrogen atom is given by the *Bohr radius*,

$$a_{\rm B} \equiv \frac{4\pi\varepsilon_0\hbar^2}{e^2 m_e} = \frac{\hbar c}{\alpha m_e c^2} = \frac{\lambda\!\!\!^{-}_{\rm C}}{\alpha} = 0.529\,\text{Å} \ . \tag{8.9}$$

The *classical electron radius* is

$$r_e \equiv \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = \frac{\alpha\hbar c}{m_e c^2} = \alpha\lambda\!\!\!^{-}_{\rm C} = 2.82 \times 10^{-5}\,\text{Å} \ . \tag{8.10}$$