Problems for HW 2

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1 HW2 Problem 1

Consider a plasma, containing a uniform background charge-density $+eN_0$, provided by positive ions, and a gas of free electrons at temperature T, with density $N(r)$. We suppose that the ions are stationary, and that the average charge density far from the origin is zero: $N(r \to \infty) = N_0$. At the origin resides an impurity ion with charge $+Q$. Its electric field will be partly cancelled by passing electrons. The Poisson equation will determine the potential Φ:

$$\nabla^2 \Phi(r) = \frac{1}{\epsilon_0} e(N(r) - N_0) - Q\,\delta^3(\vec{r}) \tag{1}$$

The average density of electrons is given by the Boltzmann factor:

$$\frac{N(r)}{N_0} = e^{e\Phi(r)/kT}. \tag{2}$$

Make the approximation of large distances from the ion: $e\Phi(r) << kT$. Find the resulting potential. The characteristic length scale of this potential is the "Debye length" or "screening length" which is important in biophysics, astrophysics, and many other areas of physics and chemistry. Electrostatic fields can penetrate only this length into a fluid containing free charges such as a plasma.

2 HW2 Problem 2

A dodecahedron is a regular polyhedron with 12 sides, each a regular pentagon. Interestingly, a cube can be inscribed in a dodecahedron, so that each of the 8 corners of the cube coincides with one of the 20 vertices of the dodecahedron. In this case, each edge of the cube coincides with a diagonal of one of the pentagonal faces. (See, for example http://www.math.umn.edu/~roberts/java.dir/JGV/cube-in-dodeca.html).

Find the work required to bring 20 point charges $+q$ from $r \to \infty$ to the corners of a dodecahedron. The length of one side of any pentagon of the dodecahedron is a.

One of the charges is replaced with a spherical shell with charge $+q$ and radius $(4/3)a$. What is the change in work?

3 HW2 Problem 3

Consider a solid conducting sphere of radius r_0 at potential ϕ_0, surrounded by a concentric, spherical conducting shell of radius R_0. Both are at potential ϕ_0. According to the Laplace Equation, the electric field will vanish between the spheres, and the inner sphere will contain zero charge. However, if the photon had mass m_γ, the inner sphere would contain charge, and a field would be present between the spheres, as we show here.

If the photon had mass, the electric potential would take the form

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{e^{-\mu_\gamma r}}{r} \tag{3}$$

where $\mu_\gamma = \bar{h} m_\gamma / c$. We assume that m_γ is nonzero, but very small; or equivalently that $1/\mu_\gamma$ is large compared with r_0 or R_0.

a) Show that if the photon has mass, then Gauss's Law becomes

$$\int_S \vec{E}\cdot\hat{n}da = \frac{Q}{\epsilon_0} - \mu_\gamma^2 \int_V \Phi d^3r, \tag{4}$$

where S is the closed surface bounding the volume V, and Q is the total charge inside S. The electric field is $\vec{E} = -\vec{\nabla}\Phi$.

b) Because the solid inner sphere is a conductor, it will be at potential Φ_0 throughout. (However, it may contain a charge distribution). On the other hand, the region between the spheres contains no charge, but the potential may vary.

Show that the potential between the spheres is given by

$$\Phi = \Phi_0 \frac{r_0}{r}\left[K_+ e^{\mu_\gamma(r-r_0)} + K_- e^{-\mu_\gamma(r-r_0)}\right] \tag{5}$$

where the constants K_+, K_- satisfy the boundary conditions

$$K_+ + K_- = 1 \tag{6}$$

$$\frac{r_0}{R_0}\left[K_+ e^{\mu_\gamma(R_0-r_0)} + K_- e^{-\mu_\gamma(R_0-r_0)}\right] = 1. \tag{7}$$

Argue that the potential has a minimum between the spheres.

c) Show that the electric field at the surface of the inner sphere is radial, with magnitude

$$E_r(r_0) = -\left.\frac{d\Phi}{dr}\right|_{r_0} = \frac{\Phi_0}{r_0}\left[1 - \mu_\gamma r_0(2K_+ - 1)\right] \tag{8}$$

d) Show that to first order in the photon mass, the charge on the inner sphere is given by:

$$\frac{Q}{4\pi\epsilon_0\, r_0}\frac{1}{\Phi_0} = 1 - \mu_\gamma r_0(2K_+ - 1) + \frac{1}{3}\mu_\gamma^2 r_0^2 \approx \frac{1}{6}\mu_\gamma^2(R_0^2 + R_0 r_0) \tag{9}$$

A constraint on the photon mass can be set in the laboratory by constructing a pair of spheres, connected with a thin wire. An alternating potential is applied to the outer sphere and the wire is monitored for a current at that frequency.