Problems for HW 3

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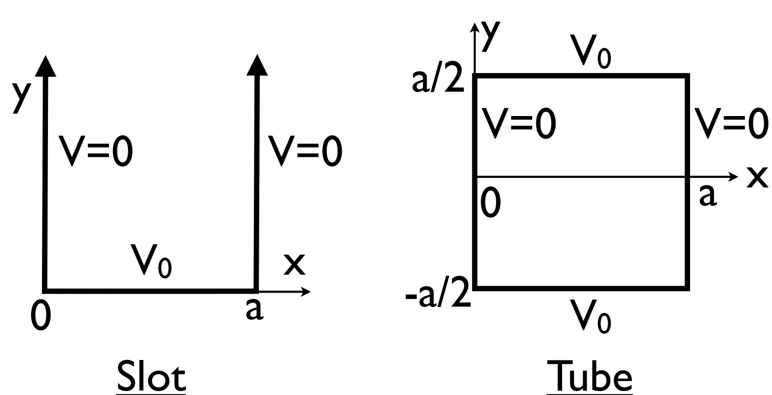
1 HW3 Problem 1

In Sec. 2.9, on p. 71, Jackson discusses the problem of the potential within a rectangular box, with a potential $V(x, y)$ specified on the one wall at $z = c$, and the other 5 walls grounded conductors. Jackson solves this equation with a Fourier sine series, as given by Eq. 2.57 (basis functions given by Eqs. 2.53, 2.54). Note that the periods of the sine waves are *twice* the lengths of the walls along x and y; this is a feature of the sine series.

In Sec 2.10, Jackson discusses a related example, a slot with semi-infinite grounded conducting walls, and a base at potential V_0. The base lies at $y = 0$ and extends a distance a toward $+x$, the walls extend toward $y \to \infty$, and the slot has infinite length in z (in other words, the potential is independent of z). He presents the solution as a sine-transform in the x-direction, with a corresponding exponential falloff in y for each term.

a) Consider a situation intermediate between the example of 2.10 and the discussion of 2.9: a rectangular tube with square cross-section. The length of each side in x and y is a, and the tube is infinitely long in z (in other words, the potential is independent of z). Suppose that the walls at $y = \pm a/2$ are at potential V_0, and the walls at $x = 0$



and $x = a$ are held at potential $V = 0$. You can solve this problem by replacing $\exp(-n\pi y/a)$ by $\cosh(n\pi y/a)/\cosh(n\pi/2)$, in the expressions of Jackson's Section 2.10.

For this problem, the boundary conditions seem to be unchanged if you subtract V_0 from the potentials on the walls, multiply by -1, and rotate the tube by 90°. (You may also translate the tube, to take into account the offset of the origin, if you wish). Is the solution identical after these operations? (Note that cosh functions plus a constant have replaced sines, and sines plus a constant have replaced cosh.) What is the value of the potential at the center of the tube?

b) A more common form of Fourier series might include both sine and cosine terms, with the lengths of sides equal to an integer number of wave periods (rather than half-periods, as in the sine series that Jackson uses):

$$f(x) = \sum_{n=-\infty}^{\infty} \tilde{A}_n \exp\left\{ i\frac{2\pi n}{a} x \right\}, \tag{1}$$

for the domain $-a/2 \le x < a/2$, and n runs over all integers. Note that if f is real, then $\tilde{A}_n = (\tilde{A}_{-n})^*$. Also note that this form takes the origin at the middle of of the slot: the base of the slot extends from $x = -a/2$ to $x = +a/2$ in Jackson's Figure 2.10.

Suppose that you describe the potential at the base of the slot in 2.10 of Jackson using the expression of Eq. 1, rather than the sine series Jackson uses. What values of n would contribute? How would the potential vary with y?

To isolate effects of the discontinuities at the boundaries, you might wish to consider more straightforward boundary conditions. For example, consider a slot with grounded walls ($V = 0$), and with the potential on the base of the slot set to:

$$V(x,\, y=0,\, z) = V_0 \cos(\pi x/a). \tag{2}$$

The slot extends over the domain $-a/2 < x < a/2$. Note that the potential falls to zero at either end, and has one maximum, at $x = 0$. We demand that $V \to 0$ for $y \to \infty$. Clearly, the solution to Laplace's Equation inside the slot is:

$$V(x,\, y,\, z) = V_0 \cos(\pi x/a) \exp(-\pi y/a). \tag{3}$$

Suppose that you, stubbornly, choose instead to use the complex Fourier series, Eq. 1, to describe this boundary condition. Is the resulting dependence on y the same as in Eq. 3?

The set of functions used in Eq. 1 is well-known to be complete on x; is it somehow *in*complete when used to solve the Laplace Equation via separation of variables? What problems are the two varieties of transforms best at solving? Note: This question is worthy of some thought.

2 HW3 Problem 2

In complex analysis, a *holomorphism* is a function that can be expressed as a convergent power series, such as:

$$\tilde{f}(\tilde{z}) = \sum_{n=0}^{\infty} \tilde{a}_n \tilde{z}^n, \tag{4}$$

where $\tilde{f}$, $\tilde{z}$ and $\tilde{a}_n$ may be complex. Usually, a function is holomorphic only over some specific "open set" in the complex plane. (Note: Here the "tilde" accent: $\tilde{}$ denotes a complex value.)

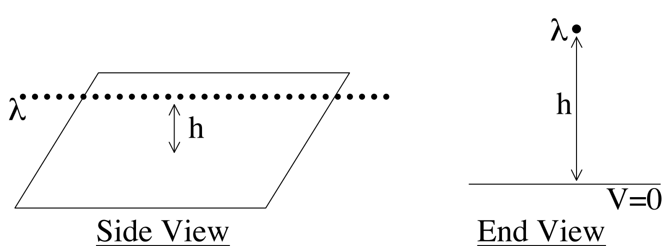
a) Prove that holomorphisms are solutions to Laplace's Equation. More precisely, if $\tilde{f}(\tilde{z})$ is a holomorphism, then let $\tilde{z} = x + iy = s\,e^{i\phi}$, and show that

$$\nabla^2 \mathrm{Re}[f(x,y)] = \nabla^2 \mathrm{Im}[\tilde{f}(x,y)] = 0. \tag{5}$$

Use the differentiability of holomorphisms, and the chain rule, to show that if $\tilde{f}$ and $\tilde{g}$ are holomorphisms, then the real and imaginary parts of $\tilde{f}(\tilde{g}(\tilde{z}))$ are solutions of Laplace's Equation. Thus, if you know one solution to Laplace Equation in 2D, this fact can be used to find others.

If $\tilde{g}$ is holomorphic over a region $s_0 < s < s_1$, $\phi_0 < \phi < \phi_1$, and $\tilde{f}$ is holomorphic everywhere, then find the region over which the solution $\tilde{f}(\tilde{g})$ is valid.

b) An infinite line charge λ resides a distance h from an infinite, grounded plane. Find the potential V_{tot}, due to the line charge and the resulting image charge on the plane. Please express your result in cylindrical coordinates, with the origin in the conducting plane at the point nearest the line charge. Set the angular coordinate so that the line charge is at $s = h$, $\phi = \pm\pi/2$, and the grounded plane is at $\phi = 0$ and $\phi = \pi$. Note that this is a solution to Laplace's Equation in a half-space, except at the location of the line charge. (Hint: You may wish to use the law of cosines: $|\vec{h} - \vec{s}| = \sqrt{h^2 + s^2 - 2hs\cos(\theta)}$, where θ is the angle between $\vec{h}$ and $\vec{s}$.)



c) Using the technique in part a, you can extend your solution V_{tot} from part b to a larger range of problems, using the function $\tilde{f}(\tilde{g}(\tilde{z})) = \tilde{f}(\tilde{x}^{\alpha})$ with values of $\alpha > 0$. For example, what does the solution look like for $\alpha = 2$? Please describe the physical situation of the charge and boundary conditions.

What if $\alpha = 1/2$? Describe the physical situation.

What happens if you use $\alpha < 0$? Are these functions all holomorphisms? (anywhere? everywhere?)

d) What is the charge per unit length of the line charge? Does it change under this transformation? You may want to use Mathematica or a similar package for this part.