NAME:

Physics 210A: Fall 2009
MIDTERM EXAM

This exam consists of 2 problems, each with 2 parts for a total of 4 parts. Each part is worth 25 points, for a total of 100 points.

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient :
$$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence :
$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl :
$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian :
$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient :
$$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence :
$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian :
$$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient :
$$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence :
$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl :
$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian :
$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

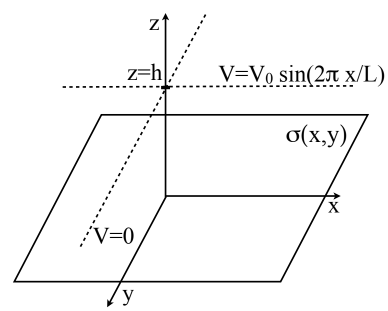
1) An *unknown* surface charge lies in the $x-y$ plane. No other charge resides anywhere else in space. The potential falls to zero far from the plane: $V \to 0$ for $z \to \pm\infty$.
The potential along a line parallel to the x-axis, above the $x-y$ plane at $z=h$, $y=0$ is given by:

$$V(x, y=0, z=h) = V_0 \sin(2\pi x/L) \tag{1}$$

The potential along the perpendicular line, parallel to the y-axis at $z=h$, $x=0$, is zero:

$$V(x=0, y, z=h) = 0. \tag{2}$$

a) Write down an expression for the potential throughout space that matches these boundary conditions. You need not derive this mathematically, if you explain why it is a solution.



Solution:

The potential

$$V(x,y,z) = V_0 \sin\left(\frac{2\pi}{L}x\right) \exp\left\{-\frac{2\pi}{L}(|z|-h)\right\} \tag{3}$$

matches the boundary conditions.

This form is taken from separation of variables for Cartesian coordinates.

Note that if charge is confined to the plane $z=0$ then this form holds in both half-spaces.

b) Is the solution you found for part a unique? If so, explain why it is unique. If not, give a different solution that matches these boundary conditions.

Solution:

The solution given in part a is *not* unique.

Note: the boundary conditions are not Dirichlet because they are not defined on a closed surface, but only along two lines. (Note that the surface charge is said to be *unknown.*) Thus, the uniqueness theorems for Dirichlet boundary conditions are not applicable. Moreover, the Green's function for such boundary conditions isn't useful in this case.

The potential

$$V(x, y, z) = V_0 \sin\left(\frac{2\pi}{L}x\right) \cos(\beta y) \exp\{-\gamma(|z| - h)\}, \qquad \text{where} \gamma = \sqrt{\beta^2 + (2\pi/L)^2} \quad (4)$$

also matches the boundary conditions.

This form is again taken from separation of variables for Cartesian coordinates.

2) Consider the electrostatic potential at point $\vec{r}$:

$$\Phi(\vec{r}) = \Phi_0 \frac{e^{-\mu|\vec{r}-\vec{R}|}}{|\vec{r}-\vec{R}|}. \tag{5}$$

The vector $\vec{R}$ is a parameter.
a) Find the charge density $\rho(\vec{r})$ responsible for this potential. (Note: You may translate the coordinate origin, as long as you remember to translate it back afterwards!)

Solution:
The charge density is given by:

$$\rho = -\epsilon_0 \nabla^2 \Phi(\vec{r}) \tag{6}$$

Introduce the translation:

$$\vec{r}_1 = \vec{r} - \vec{R}. \tag{7}$$

Then,

$$\Phi(\vec{r}_1) = \Phi_0 \frac{e^{-\mu|\vec{r}_1|}}{|\vec{r}_1|}. \tag{8}$$

Spherical coordinates centered at the origin of $\vec{r}_1$ are convenient; then, the potential depends only on the distance from that point $r_1 = |\vec{r}_1|$.
Then, *as long as* $r_1 \neq 0$,

$$\nabla^2 \Phi(r_1) = \frac{1}{r_1^2}\frac{\partial}{\partial r_1} r_1^2 \frac{\partial}{\partial r_1}\Phi(r_1) \tag{9}$$

$$= \frac{1}{r_1}\frac{\partial^2}{\partial {r_1}^2}\, r_1\, \Phi(r_1), \tag{10}$$

where Jackson gives the second, convenient form inside the back cover. (This was also mentioned in class). This gives, for the charge density at $r_1 \neq 0$,

$$\rho = -\epsilon_0 \Phi_0\, \mu^2\, \frac{e^{-\mu r_1|}}{r_1} = -\mu^2 \Phi(r_1) \tag{11}$$

where the last equality is convenient!
The special case $r_1 = 0$ requires special treatment. Note that the above expressions for ∇^2 are not defined there. For $r_1 \to 0$, the potential $\Phi(r_1) \to \Phi_0/r_1$. This is the potential of a point charge at the origin (as explored in problems on Debye shielding and a massive photon). The Laplacian is then given by

$$\nabla^2|_{r_1=0}\Phi(r_1) = -4\pi\Phi_0\delta^3(\vec{r}_1) \tag{12}$$

(see argument in Jackson using the Divergence Theorem to evaluate this).

Combining the two contributions to ρ we find:

$$\rho(\vec{r}) = 4\pi\epsilon_0\Phi_0\delta^3(\vec{r} - \vec{R}) - \epsilon_0\mu^2\Phi(\vec{r}) \tag{13}$$

This is a point charge at $\vec{R}$, with a surrounding neutralizing halo. The net charge is zero, as can be seen by integrating this expression or observing that the term in the potential proportional to $1/r$ vanishes.

b) Now consider the electrostatic potential

$$\Phi(\vec{r}) = p_0 \, e^{-\mu r} \left(\frac{1}{r^2} + \frac{\mu}{r} \right) \cos\theta. \tag{14}$$

Here, (r, θ, ϕ) are spherical coordinates. (Note that here I have corrected the $-$ sign inside the parentheses, as also announced in class.)
Describe the charge density; you need not work this out mathematically if you briefly explain your argument. What is the *net* dipole moment of the underlying charge density?

Solution:

This potential is the *difference* of 2 copies of the potential from part a. The copies are offset by a separation $d\hat{z}$, and Φ_0 is scaled to Φ_0/d; these are differenced in the limit $d \to 0$. This is precisely the construction used to make a physical dipole into a point dipole; here, the construction is carried out with neutralizing halos (or, if you prefer, with a massive photon).

I introduce the notation Φ_a for the potential used in part a, and Φ_b for the potential introduced above. Mathematically, with $\vec{R} = 0$,

$$\frac{\partial}{\partial z} \Phi_a = \Phi_0 \left(-\mu - \frac{1}{r} \right) \left(\frac{e^{-\mu r}}{r} \right) \frac{\partial r}{\partial z} \tag{15}$$

and also:

$$\frac{\partial r}{\partial z} = \frac{1}{2} \frac{1}{r} 2z = \frac{z}{r} = \cos\theta \tag{16}$$

So,

$$\frac{\partial}{\partial z} \Phi_a = \Phi_0 \left\{ e^{-\mu r} \left(\frac{\mu}{r} - \frac{1}{r^2} \right) \cos\theta \right\} \tag{17}$$

and consequently,

$$\Phi_b = \left(\frac{-p_0}{\Phi_0} \right) \frac{\partial}{\partial z} \Phi_a \tag{18}$$

Then, using the definition of derivative,

$$\Phi_b = \left(\frac{-p_0}{\Phi_0} \right) \lim_{d \to 0} \frac{1}{d} \Big(\Phi_a(\vec{r} - d/2 \; \hat{z}) - \Phi_a(\vec{r} + d/2 \; \hat{z}) \Big) \tag{19}$$

as promised above. (Note: "net" dipole moment is a hint of the delta-function, and the offset by $\vec{R}$ in 2a might foreshadow the above construction).

The charge density is a point dipole at the origin, surrounded by a neutralizing dipole halo.

The *net* dipole moment of the charge distribution is zero, as can be seen from the fact that the net charge of each of the two contributing charge distributions is zero, as mentioned above. Alternatively, you could calculate the charge distribution and integrate $\vec{r}'\rho$. This is fairly tedious.

Finally, and most likely just for fun, you could evaluate the dipole moment by finding the term in the potential that is proportional to $1/r^2$ at large r. In this connection, you might recall that the Taylor series of the function $e^{-1/u}$ about $u = 0$ is zero, for all powers of u. In another form,

$$\lim_{r\to\infty} (\mu r)^n e^{-\mu r} = 0, \qquad \text{for all } n \tag{20}$$

This interesting and always-astounding fact can be proved using L'Hospital's rule and induction; or, for our purposes (the lowest few powers) by direct calculation. The particular fact we need is

$$\lim_{r\to\infty} r^2 \Phi_b = 0 \tag{21}$$

which follows directly from the previous equation and the definition of Φ_b above. More generally, for our potential Φ_b, all multipole moments are zero. The same is true, of course, for Φ_a.