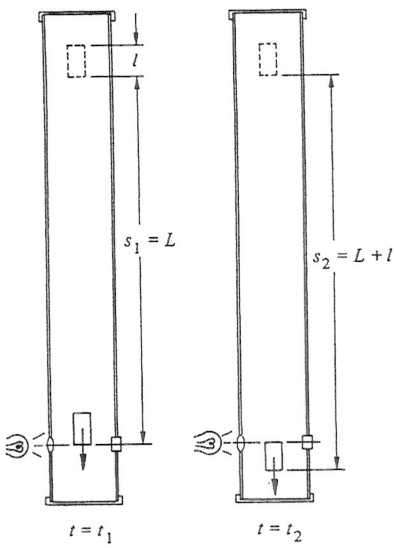
Note 1.1 Mathematical Approximation Methods

Occasionally in the course of solving a problem in physics you may find that you have become so involved with the mathematics that the physics is totally obscured. In such cases, it is worth stepping back for a moment to see if you cannot sidestep the mathematics by using simple approximate expressions instead of exact but complicated formulas. If you have not yet acquired the knack of using approximations, you may feel that there is something essentially wrong with the procedure of substituting inexact results for exact ones. However, this is not really the case, as the following example illustrates.



Suppose that a physicist is studying the free fall of bodies in vacuum, using a tall vertical evacuated tube. The timing apparatus is turned on when the falling body interrupts a thin horizontal ray of light located a distance L below the initial position. By measuring how long the body takes to pass through the light beam, the physicist hopes to determine the local value of g, the acceleration due to gravity. The falling body in the experiment has a height l.

For a freely falling body starting from rest, the distance s traveled in time t is

$$s = \tfrac{1}{2}gt^2,$$

which gives

$$t = \sqrt{\frac{2}{g}}\sqrt{s}.$$

The time interval $t_2 - t_1$ required for the body to fall from $s_1 = L$ centimeters to $s_2 = (L + l)$ centimeters is

$$t_2 - t_1 = \sqrt{\frac{2}{g}}(\sqrt{s_2} - \sqrt{s_1})$$
$$= \sqrt{\frac{2}{g}}(\sqrt{L + l} - \sqrt{L}).$$

If $t_2 - t_1$ is measured experimentally, g is given by

$$g = 2\left(\frac{\sqrt{L + l} - \sqrt{L}}{(t_2 - t_1)}\right)^2$$

This formula is exact under the stated conditions, but it may not be the most useful expression for our purposes.

Consider the factor

$$\sqrt{L + l} - \sqrt{L}.$$

In practice, L will be large compared with l (typical values might be $L = 100$ cm, $l = 1$ cm). Our factor is the small difference between two large numbers and is hard to evaluate accurately by using a slide rule or ordinary mathematical tables. Here is a simple approach, known as the method of power series expansion, which enables us to evaluate the factor to any accuracy we please. As we shall discuss formally later in this Note, the quantity $\sqrt{1+x}$ can be written in the series form

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \cdots$$

for $-1 < x < 1$. Furthermore, if we cut off the series at some point, the error we incur by this approximation is of the order of the first neglected term. We can put the factor in a form suitable for expansion by first extracting $\sqrt{L}$:

$$\sqrt{L+l} - \sqrt{L} = \sqrt{L}\left(\sqrt{1+\frac{l}{L}} - 1\right).$$

The dimensionless ratio l/L plays the part of x in our expansion. Expanding $\sqrt{1 + l/L}$ in the series form gives

$$\sqrt{L}\left(\sqrt{1+\frac{l}{L}} - 1\right) = \sqrt{L}\left[1 + \frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 + \cdots - 1\right]$$

$$= \sqrt{L}\left[\frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 + \cdots\right].$$

We see that if l/L is much smaller than 1, the successive terms decrease rapidly. The first term in the bracket, $\frac{1}{2}(l/L)$, is the largest term, and extracting it from the bracket yields

$$\sqrt{L+l} - \sqrt{L} = \sqrt{L}\,\frac{1}{2}\left(\frac{l}{L}\right)\left[1 - \frac{1}{4}\left(\frac{l}{L}\right) + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \cdots\right]$$

$$= \frac{l}{2\sqrt{L}}\left[1 - \frac{1}{4}\left(\frac{l}{L}\right) + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \cdots\right].$$

Our expansion is now in its final and most useful form. The first factor, $l/(2\sqrt{L})$, gives the dominant behavior and is a useful first approximation. Furthermore, writing the series as we have, with leading term 1, shows clearly the contributions of the successive powers of l/L. For example, if $l/L = 0.01$, the term $\frac{1}{8}(l/L)^2 = 1.2 \times 10^{-5}$ and we make a fractional error of about 1 part in 10^5 by retaining only the preceding terms. In many cases this accuracy is more than enough. For instance, if the time interval $t_2 - t_1$ in the falling body experiment can be measured to only 1 part in 1,000, we gain nothing by evaluating $\sqrt{L+l} - \sqrt{L}$ to greater accuracy than this. On the other hand, if we require greater accuracy, we can easily tell how many terms of the series should be retained.

Practicing physicists make mathematical approximations freely (when justified) and have no compunctions about discarding negligible terms. The ability to do this often makes the difference between being stymied by impenetrable algebra and arithmetic and successfully solving a problem.

Furthermore, series approximations often allow us to simplify complicated algebraic expressions to bring out the essential physical behavior.

Here are some helpful methods for making mathematical approximations.

1 THE BINOMIAL SERIES

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$$

This series is valid for $-1 < x < 1$, and for any value of n. (If n is an integer, the series terminates, the last term being x^n.) The series is exact; the approximation enters when we truncate it. For $n = \frac{1}{2}$, as in our example,

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \cdots \qquad -1 < x < 1.$$

If we need accuracy only to $O(x^2)$ (order of x^2), we have

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + O(x^3),$$

where the term $O(x^3)$ indicates that terms of order x^3 and higher are not being considered. As a rule of thumb, the error is approximately the size of the first term dropped.

The series can also be applied if $|x| > 1$ as follows:

$$(1+x)^n = x^n\left(1+\frac{1}{x}\right)^n = x^n\left[1 + n\frac{1}{x} + \frac{n(n-1)}{2!}\left(\frac{1}{x}\right)^2 + \cdots\right].$$

Examples:

1. $\dfrac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \qquad -1 < x < 1$

2. $\dfrac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \qquad -1 < x < 1$

3. $(1{,}001)^{\frac{1}{3}} = (1{,}000 + 1)^{\frac{1}{3}} = 1{,}000^{\frac{1}{3}}(1 + 0.001)^{\frac{1}{3}} = 10[1 + 0.001(\tfrac{1}{3}) + \cdots] \approx 10(1.0003) = 10.003$

4. $2 - \dfrac{1}{\sqrt{1+x}} - \dfrac{1}{\sqrt{1-x}}$: for small x, this expression is zero to first

approximation. However, this approximation may not be adequate. Using the binomial series, we have

$$2 - \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1-x}} = 2 - (1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots) - (1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \cdots)$$
$$= -\tfrac{3}{4}x^2.$$

Notice that the terms linear in x also cancel. To obtain a nonvanishing result we had to go to a high enough order, in this case to order x^2. It is clear that for a correct result we have to expand all terms to the same order.

2 TAYLOR'S SERIES[1]

Analogous to the binomial series, we can try to represent an arbitrary function $f(x)$ by a power series in x:

$$f(x) = a_0 + a_1x + a_2x + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

For $x = 0$ we must have

$$f(0) = a_0.$$

Assuming for the moment that it is permissible to differentiate, we have

$$\frac{df}{dx} = f'(x) = a_1 + 2a_2x + \cdots$$

Evaluating at $x = 0$ we have

$$a_1 = f'(x)\Big|_{x=0}.$$

Continuing this process, we find

$$a_k = \frac{1}{k!} f^{(k)}(x)\Big|_{x=0},$$

where $f^{(k)}(x)$ is the kth derivative of $f(x)$. For the sake of a less cumbersome notation, we often write $f^{(k)}(0)$ to stand for $f^{(k)}(x)\Big|_{x=0}$; but bear in mind that $f^{(k)}(0)$ means that we should differentiate $f(x)$ k times and *then* set x equal to 0.

The power series for $f(x)$, known as a *Taylor series,* can then be expressed formally as

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots.$$

This series, if it converges, allows us to find good approximations to $f(x)$ for small values of x (that is, for values of x near zero). Generalizing,

$$f(a + x) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \cdots$$

[1] Taylor's series is discussed in most elementary calculus texts. See the list at the end of the chapter.

gives us the behavior of the function in the neighborhood of the point a. An alternative form for this expression is

$$f(t) = f(a) + f'(a)(t - a) + f''(a)\frac{(t - a)^2}{2!} + \cdots.$$

Our formal manipulations are valid only if the series converges. The range of convergence of a Taylor series may be $-\infty < x < \infty$ for some functions (such as e^x) but quite limited for other functions. (The binomial series converges only if $-1 < x < 1$.) The range of convergence is hard to find without considering functions of a complex variable, and we shall avoid these questions by simply assuming that we are dealing with simple functions for which the range of convergence is either infinite or is readily apparent. Here are some examples:

a. The Trigonometric Functions

Let $f(x) = \sin x$, and expand about $x = 0$.

$$f(0) = \sin(0) = 0$$
$$f'(0) = \cos(0) = 1$$
$$f''(0) = -\sin(0) = 0$$
$$f'''(0) = -\cos(0) = -1, \quad \text{etc.}$$

Hence

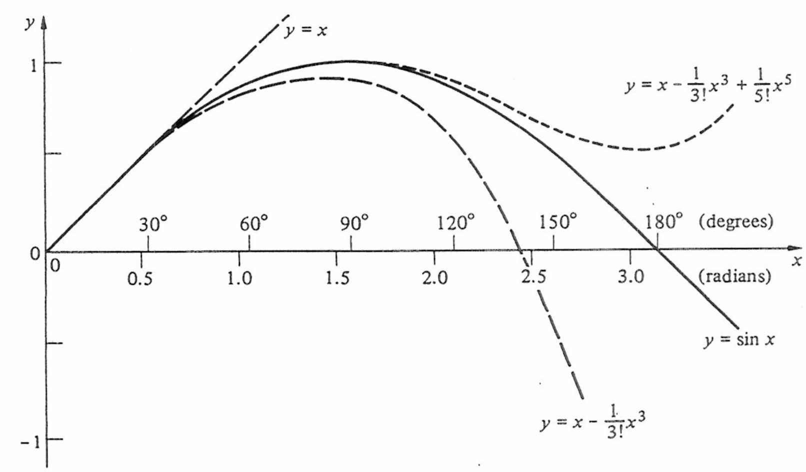
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots.$$

Similarly

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

These expansions converge for all values of x but are particularly useful for small values of x. To $O(x^2)$, $\sin x = x$, $\cos x = 1 - x^2/2$.

The figure below compares the exact value for $\sin x$ with a Taylor series in which successively higher terms are included. Note how each



term increases the range over which the series is accurate. If an infinite number of terms are included, the Taylor series represents the function accurately everywhere.

b. The Binomial Series

We can derive the binomial series introduced in the last section by letting

$$f(x) = (1 + x)^n.$$

Then

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= n(1 + 0)^n = n \\ f''(0) &= n(n - 1) \\ f^{(k)}(0) &= n(n - 1)(n - 2) \cdots (n - k + 1) \end{aligned}$$

$$(1 + x)^n = 1 + nx + \frac{1}{2!} n(n - 1)x^2 + \cdots + \cdots \frac{n(n - 1) \cdots (n - k + 1)}{k!} x^k + \cdots$$

c. The Exponential Function

If we let $f(x) = e^x$, we have $f'(x) = f(x)$, by the definition of the exponential function. Similarly $f^{(k)}(x) = f(x)$. Since $f(0) = e^0 = 1$, we have

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots.$$

This series converges for all values of x.

A useful result from the theory of the Taylor series is that if the series converges at all, it represents the function so well that we are allowed to differentiate or integrate the series any number of times. For example,

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots\right) \\ &= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \\ &= \cos x. \end{aligned}$$

Furthermore, the Taylor series for the product of two functions is the product of the individual series:

$$\begin{aligned} \sin x \cos x &= \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots\right)\left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots\right) \\ &= x - \left(\frac{1}{3!} + \frac{1}{2!}\right) x^3 + \left(\frac{1}{4!} + \frac{1}{3!2!} + \frac{1}{5!}\right) x^5 + \cdots \end{aligned}$$

$$= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} + \cdots$$

$$= \frac{1}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots\right]$$

$$= \frac{1}{2}[\sin(2x)].$$

The Taylor series sometimes comes in handy in the evaluation of integrals. To estimate

$$\int_1^{1.1} \frac{e^z}{z}\, dz,$$

let $z = 1 + x$. We then have

$$\int_1^{1.1} \frac{e^z}{z}\, dz = \int_0^{0.1} \frac{e^{(1+x)}}{1+x}\, dx$$

$$= (e)\int_0^{0.1} \frac{e^x}{1+x}\, dx$$

$$\approx (e)\int_0^{0.1} \frac{(1+x)}{(1+x)}\, dx$$

$$\approx 0.1e.$$

The approximation should be better than 1 part in 100 or so, for x always lies in the interval $0 \le x \le 0.1$. In this range, $e^x \approx 1 + x$ is a good approximation to two or three significant figures.

3 DIFFERENTIALS

Consider $f(x)$, a function of the independent variable x. Often we need to have a simple approximation for the change in $f(x)$ when x is changed to $x + \Delta x$. Let us denote the change by $\Delta f = f(x + \Delta x) - f(x)$. It is natural to turn to the Taylor series. Expanding the Taylor series for $f(x)$ about the point x gives

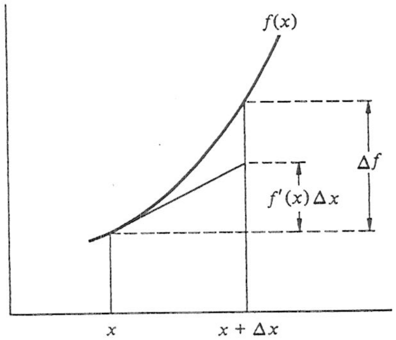
$$f(x + \Delta x) = f(x) + f'(x)\,\Delta x + \frac{1}{2!} f''(x)\,\Delta x^2 + \cdots,$$

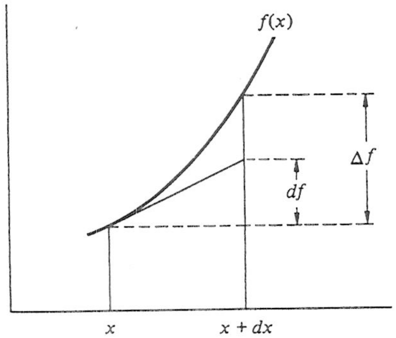
where, for example, $f'(x)$ stands for df/dx evaluated at the point x. Omitting terms of order $(\Delta x)^2$ and higher yields the simple linear approximation

$$\Delta f = f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x.$$

This approximation becomes increasingly accurate the smaller the size of Δx. However, for finite values of Δx, the expression

$$\Delta f \approx f'(x)\,\Delta x$$





has to be considered to be an approximation. The graph at left shows a comparison of $\Delta f \equiv f(x + \Delta x) - f(x)$ with the linear extrapolation $f'(x)\,\Delta x$. It is apparent that Δf, the actual change in $f(x)$ as x is changed, is generally not exactly equal to Δf for finite Δx.

As a matter of notation, we use the symbol dx to stand for Δx, the increment in x. dx is known as the *differential* of x; it can be as large or small as we please. We *define* df, the differential of f, by

$$df \equiv f'(x)\,dx.$$

This notation is illustrated in the lower drawing. Note that dx and Δx are used interchangeably. On the other hand, df and Δf are different quantities. df is a differential defined by $df = f'(x)\,dx$, whereas Δf is the actual change $f(x + dx) - f(x)$. Nevertheless, when the linear approximation is justified in a problem, we often use df to represent Δf. We can always do this when eventually a limit will be taken. Here are some examples.

1. $d(\sin\theta) = \cos\theta\,d\theta.$
2. $d(xe^{x^2}) = (e^{x^2} + 2x^2e^{x^2})\,dx.$
3. Let V be the volume of a sphere of radius r:

$$V = \tfrac{4}{3}\pi r^3$$
$$dV = 4\pi r^2\,dr.$$

4. What is the fractional increase in the volume of the earth if its average radius, 6.4×10^6 m, increases by 1 m?

$$\frac{dV}{V} = \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3}$$
$$= 3\frac{dr}{r}$$
$$= \frac{3}{6.4 \times 10^6} = 4.7 \times 10^{-7}.$$

One common use of differentials is in changing the variable of integration. For instance, consider the integral

$$\int_a^b xe^{x^2}\,dx.$$

A useful substitution is $t = x^2$. The procedure is first to solve for x in terms of t,

$$x = \sqrt{t},$$

and then to take differentials:

$$dx = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt.$$

This result is exact, since we are effectively taking the limit. The original integral can now be written in terms of t:

$$\int_a^b xe^{x^2}\,dx = \int_{t_1}^{t_2} \sqrt{t}\, e^t \left(\frac{1}{2}\frac{1}{\sqrt{t}}\,dt\right) = \tfrac{1}{2}\int_{t_1}^{t_2} e^t\,dt$$

$$= \tfrac{1}{2}(e^{t_2} - e^{t_1}),$$

where $t_1 = a^2$ and $t_2 = b^2$.

Some References to Calculus Texts

A very popular textbook is G. B. Thomas, Jr., "Calculus and Analytic Geometry," 4th ed., Addison-Wesley Publishing Company, Inc., Reading, Mass.

The following introductory texts in calculus are also widely used:
M. H. Protter and C. B. Morrey, "Calculus with Analytic Geometry," Addison-Wesley Publishing Company, Inc., Reading, Mass.
A. E. Taylor, "Calculus with Analytic Geometry," Prentice-Hall, Inc., Englewood Cliffs, N.J.
R. E. Johnson and E. L. Keokemeister, "Calculus With Analytic Geometry," Allyn and Bacon, Inc., Boston.

A highly regarded advanced calculus text is R. Courant, "Differential and Integral Calculus," Interscience Publishing, Inc., New York.

If you need to review calculus, you may find the following helpful: Daniel Kleppner and Norman Ramsey, "Quick Calculus," John Wiley & Sons, Inc., New York.

Problems

1.1 Given two vectors, $\mathbf{A} = (2\hat{\imath} - 3\hat{\jmath} + 7\hat{\mathbf{k}})$ and $\mathbf{B} = (5\hat{\imath} + \hat{\jmath} + 2\hat{\mathbf{k}})$, find: (*a*) $\mathbf{A} + \mathbf{B}$; (*b*) $\mathbf{A} - \mathbf{B}$; (c) $\mathbf{A} \cdot \mathbf{B}$; (*d*) $\mathbf{A} \times \mathbf{B}$.

Ans. (*a*) $7\hat{\imath} - 2\hat{\jmath} + 9\hat{\mathbf{k}}$; (c) 21

1.2 Find the cosine of the angle between

$\mathbf{A} = (3\hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}})$ and $\mathbf{B} = (-2\hat{\imath} - 3\hat{\jmath} - \hat{\mathbf{k}})$.

Ans. -0.805

1.3 The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosine of the angles between the vector and the x, y, and z axes are usually called, in turn α, β, and γ. Prove that $\alpha^2 + \beta^2 + \gamma^2 = 1$, using either geometry or vector algebra.

1.4 Show that if $|\mathbf{A} - \mathbf{B}| = |\mathbf{A} + \mathbf{B}|$, then $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

1.5 Prove that the diagonals of an equilateral parallelogram are perpendicular.

1.6 Prove the law of sines using the cross product. It should only take a couple of lines. (*Hint*: Consider the area of a triangle formed by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$.)