## HRK 20.8

Denote the speed of the S waves as $v_s$ and the speed of the P waves as $v_p$. The time taken to reach the surface is $t_s$ for an S wave, and $t_p$ for a P wave. If the earthquake occurred $d$ m below the surface then

$$\begin{aligned}
v_s = \frac{d}{t_s} \quad \text{and} \quad v_p &= \frac{d}{t_p} \\
t_s - t_p &= \frac{d}{v_s} - \frac{d}{v_p} \\
180\ s &= d\left(\frac{1}{4500\ ms^{-1}} - \frac{1}{8200\ ms^{-1}}\right) \\
\Rightarrow d &= 1.795 \times 10^6\ m
\end{aligned}$$

## HRK 20.11

Compare the form of the sound wave given to the general form

$$\begin{aligned}
\text{Given:} \quad \Delta p(x,t) &= (1.48\ Pa)\sin\big((1.07\pi)x - (334\pi)t\big) \\
\text{General:} \quad \Delta p(x,t) &= \Delta p_m \sin(kx - \omega t)
\end{aligned}$$

Comparison easily tells you

## 20.11 (a)

$$\text{Pressure amplitude: } \Delta p_m = 1.48\ Pa$$

## 20.11 (b)

$$\begin{aligned}
\omega &= 334\pi \\
2\pi\nu &= 334\pi \\
\nu &= \frac{334\pi}{2\pi} = 167\ Hz.
\end{aligned}$$

## 20.11 (c)

$$\begin{aligned}
k &= 1.07\pi \\
\frac{2\pi}{\lambda} &= 1.07\pi \\
\lambda &= \frac{2}{1.07} = 1.87\ m
\end{aligned}$$

## 20.11 (d)

$$\begin{aligned}
v &= \frac{\omega}{k} \\
v &= \frac{334\pi}{1.07\pi} = 312\ ms^{-1}
\end{aligned}$$

## HRK 20.13

Recall that for spherical waves, the wave intensity varies inversely as the square of the distance

$$I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

In our case the intensity is $1.97 \times 10^{-8}$ $W$ at a distance 42.5 $m$ from the source

$$\begin{aligned} 1.97 \times 10^{-8} &= \frac{P}{4\pi(42.5)^2} \\ P &= 4.47\ W \end{aligned}$$

## HRK 20.14

**Method 1: Most Direct Route:**

The key relationships we need to bear in mind are summarized as

$$\frac{\Delta\rho}{\rho} = \frac{\Delta p}{B} = -\frac{\partial s}{\partial x}$$

The second equality can be re-arranged to say

$$\Delta p = -B\frac{\partial s}{\partial x}$$

For a sinusoidal wave we have

$$\begin{aligned} s(x,t) &= s_m \cos(kx - \omega t) \\ \Rightarrow -B\frac{\partial s}{\partial x} &= Bks_m \sin(kx - \omega t) = \Delta p \end{aligned}$$

Now replace the bulk modulus $B$ using $v = \sqrt{\frac{B}{\rho}}$ to get

$$\begin{aligned} \Delta p = & \underbrace{(k\rho v^2 s_m)}_{\text{Pressure Amplitude}} \sin(kx - \omega t) \\ \Delta p = & \underbrace{\Delta p_m}_{\text{Pressure Amplitude}} \sin(kx - \omega t) \end{aligned}$$

In order to find the relationship between intensity, $I$, and displacement amplitude, $s_m$, first recall that

$$\begin{aligned} I &= \Delta p \frac{\partial s}{\partial t} \\ \Rightarrow I &= \Delta p\ \omega s_m \sin(kx - \omega t) \end{aligned}$$

Now for a time-averaged sinusoidal wave we have

$$\overline{I} = \frac{1}{2}\Delta p_m \omega s_m.$$

Replace the $\Delta p_m$ term, which is a quantity we are not given in the question, with $(k\rho v^2 s_m)$

$$\begin{aligned} \overline{I} &= \frac{1}{2}(k\rho v^2 s_m)\omega s_m. \\ \overline{I} &= \frac{1}{2}k\rho v^2 \omega s_m^2 \\ \overline{I} &= \frac{1}{2}\rho v \omega^2 s_m^2 \qquad \text{using } k = \frac{\omega}{v} \end{aligned}$$

which can be re-arranged to say

$$\begin{aligned} s_m^2 &= \frac{2\overline{I}}{\rho v \omega^2} \\ s_m &= \sqrt{\frac{2\overline{I}}{\rho v \omega^2}} \\ s_m &= \sqrt{\frac{2(1.13\times10^{-6})}{(1.21)(343)(2\pi(313))^2}} \\ s_m &= 3.75\times10^{-8}\ m \end{aligned}$$

**Method 2: Manipulating relevant equations from HRK:**

Recall that the expression for time-average intensity (in the pressure amplitude picture) is given by

$$\overline{I} = \frac{(\Delta p_m)^2}{2\rho v}.$$

We will first find the value for $\Delta p_m$ and then convert to the displacement amplitude picture using the relation

$$\Delta p_m = k\rho v^2 s_m$$

$$\begin{aligned} \overline{I} &= \frac{(\Delta p_m)^2}{2\rho v} \\ \Delta p_m &= \sqrt{2\rho v I} \\ \Delta p_m &= \sqrt{2(1.21)(343)(1.13\times10^{-6})} \\ \Delta p_m &= 3.06\times10^{-2} N/m^2 \end{aligned}$$

where in the second last line I substituted in the given value for $I$ and standard density and speed-of-sound values for air.

The final step is to find the displacement amplitude $s_m$

$$\begin{aligned} \Delta p_m &= k\rho v^2 s_m \\ s_m &= \frac{\Delta p_m}{k\rho v^2} \\ s_m &= \frac{3.06\times10^{-2}}{k(1.21)(343)^2} \qquad \text{where } v = \frac{\omega}{k} \Rightarrow k = \frac{\omega}{v} = \frac{2\pi\nu}{v} = \frac{2\pi(313)}{343} = 5.73 \\ s_m &= \frac{3.06\times10^{-2}}{(5.73)(1.21)(343)^2} \\ s_m &= 3.75\times10^{-8}\ m \end{aligned}$$

We could have found this result more neatly by deriving the equation which relates intensity and $s_m$ i.e.

$$\overline{I} = \frac{(\Delta p_m)^2}{2\rho v} \quad \text{and} \quad \Delta p_m = k\rho v^2 s_m$$

$$\begin{aligned} \overline{I} &= \frac{k^2\rho^2 v^4 s_m^2}{2\rho v} \\ \overline{I} &= \frac{k^2 \rho v^3 s_m^2}{2} \\ \overline{I} &= \frac{1}{2}\rho v \omega^2 s_m^2 \qquad (\text{using} \quad v^2 = \frac{\omega^2}{k^2}) \\ \overline{I} &= \frac{1}{2}\rho v (\omega s_m)^2 \end{aligned}$$

Plug values directly into this to obtain the same value as previously

$$\begin{aligned}\bar{I} &= \frac{1}{2}\rho v(\omega s_m)^2 \\ s_m &= \sqrt{\frac{2\bar{I}}{\rho v \omega^2}} \\ s_m &= \sqrt{\frac{2(1.13\times 10^{-6})}{(1.21)(343)(2\pi(313))^2}} \\ s_m &= 3.75\times 10^{-8}\ m\end{aligned}$$

## HRK 20.23

The location of the point $B$ tells us how far the sound wave has to travel in moving through the right hand portion of the apparatus. For some position $B_{min}$ the sound intensity detected at the opening is a minimum, and for a position $B_{max}$ the sound intensity detected is a mximum. The piece of information given to us in the problem is that

$$|B_{max} - B_{min}| = 1.65\times 10^{-2} m$$

Every centimeter that $B$ is increased adds 1cm to the path before the tube bends and also 1cm to the path after the bend. This reasoning tells us that the difference in distance traveled between sound waves when $B = B_{min}$ and when $B = B_{max}$ is

$$\text{Change in distance traveled on right: } = 2|B_{max} - B_{min}| = 3.3\times 10^{-2} m$$

Since the path on the left side of the apparatus stays constant, and since there are no other maximum/minimum amplitude locations between $B_{min}$ and $B_{max}$, we know that the change in distance traveled must correspond to half a wavelength of the sound wave

$$\begin{aligned}\text{Change in distance traveled on right: } 3.3\times 10^{-2} m &= \frac{1}{2}\lambda \\ \lambda = 6.6\times 10^{-2} m\end{aligned}$$

### 20.23(a)

Assuming the speed of sound in the tube is $343 m/s$ (which is certainly not exact) we can find the frequency of the sound

$$\begin{aligned}v &= \nu\lambda \\ \nu &= \frac{v}{\lambda} \\ \nu &= \frac{343}{6.6\times 10^{-2}} \\ \nu &= 5196\ Hz = 5.2\ kHz\end{aligned}$$

### 20.23(b)

Recall from the previous chapter, as well as this one, that for all waves

$$\text{Intensity} \propto (\text{Amplitude})^2$$

We are told the ratio of the intensities for the the two configurations of the tube (i.e. when intensity is a minimum at point $D$ and when intensity is a maximum) so it easy to work out the ratio of the amplitudes for these two configurations.

$$\Rightarrow \frac{I_{max}}{I_{min}} = \frac{\left[\Delta p_m\right]^2_{max}}{\left[\Delta p_m\right]^2_{min}}$$

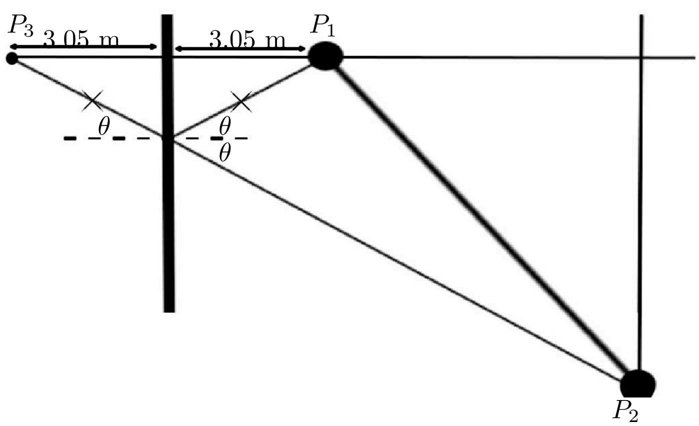
Since we are told $I_{min} = 10 \times 10^{-6}$ and $I_{max} = 90 \times 10^{-6}$ then

$$\begin{aligned} 9 &= \frac{\left[\Delta p_m\right]^2_{max}}{\left[\Delta p_m\right]^2_{min}} \\ 3 &= \frac{\left[\Delta p_m\right]_{max}}{\left[\Delta p_m\right]_{min}} \\ \left[\Delta p_m\right]_{min} &= \frac{1}{3}\left[\Delta p_m\right]_{max} \end{aligned}$$

**20.23(c)**

Constructive and destructive interference, with the waves that have traveled on the left path of the apparatus, causes a difference in amplitude at the point $D$ (the detector).

## HRK 20.32




There will be interference at point $P_2$ due to the superposition of two waves (i) the wave which travels directly form $P_1$ to $P_2$ and (ii) the wave which reflects off the wall and then travels to $P_2$. We need to find out what extra distance the second wave has traveled. Look at the picture and you can see that the distance from $P_1$ to $P_2$ via the reflection is the same as the distance between $P_2$ and a point I have called $P_3$. **Note:** As with most geometrical problems, there is not only one way to find the length of the path for reflected waves – you could, for example, use trigonometry to find a value for how far down the wall the reflected wave bounces.

$$\begin{aligned} \text{Direct path length: } |P_1 - P_2| &= \sqrt{24.4^2 + 15.2^2} = 28.747\ m \\ \text{Reflected path length: } |P_3 - P_2| &= \sqrt{(24.4 + 2(3.05))^2 + 15.2^2} = 34.077\ m \\ \text{Path Difference: } & \quad 34.077 - 28.747 = 5.33\ m \end{aligned}$$

In order to maximize sound intensity at point $P_2$ we want constructive interference to occur there. The condition for this to happen is that path difference is some integer multiple of the wavelength

$$\begin{aligned} 5.33 &= n\lambda \qquad (n = 1, 2, \ldots) \\ \lambda &= \frac{5.33}{n} \end{aligned}$$

Since $v = \nu\lambda$ and the speed of sound in air is $343m/s$ we have

$$\begin{aligned}\lambda &= \frac{343}{\nu} \\ \text{but } \lambda &= \frac{5.33}{n} \\ \frac{343}{\nu} &= \frac{5.33}{n} \\ \nu &= n\frac{343}{5.33}\end{aligned}$$

The two lowest frequencies that will gives constructive interference are found by plugging $n = 1$ and $n = 2$ into the above equation

$$\text{Lowest: } \nu = \frac{343}{5.33} = 64.35\ Hz$$
$$\text{Second Lowest: } \nu = 2\frac{343}{5.33} = 128.7\ Hz$$

## HRK 20.38

The well is like a tube that is closed at one end. We can't just use $v = 343\ m/s$ here, as we are given information about the conditions inside the well. We find

$$\begin{aligned}v &= \sqrt{\frac{B}{\rho}} \\ v &= \sqrt{\frac{1.41 \times 10^5}{1.21}} \\ v &= 341.4\ m/s\end{aligned}$$

We know that the general expression for frequency of a wave in a resonant mode (in a tube with one closed end) is given by

$$\nu = n\frac{v}{4L} \qquad n = 1, 3, 5 \ldots$$

where $L$ is the distance from the entrance of the tube to the closed end. If you cannot remember this expression it is easy to derive it by drawing a sketch. For the fundamental resonant mode we can picture a node at the entrance to the tube, and the first antinode at the closed end (in our case the surface of the water). We know that a wave goes from a minimum (node) to a maximum (antinode) in one-quarter of a wavelength and so $L = \frac{\lambda}{4}$ or, using $v = \nu\lambda$ we get $L = \frac{v}{4\nu}$ .

We are told the lowest frequency of a resonant mode (which corresponds to $n = 1$) is 7.2 Hz.

$$\begin{aligned}7.2 &= \frac{341.4}{4L} \\ L &= \frac{341.4}{4(7.2)} \\ L &= 11.85\ m\end{aligned}$$

## HRK 20.42

In the usual picture of tubes and nodes and antinodes, (i) the pressure at a closed end is a pressure anti-node, since the pressure there can vary with its maximum amplitude (ii) the pressure at an open end is a pressure node since it must be equal to the ambient pressure outside the tube.

Recall that a pressure node is a displacement antinode, a pressure antinode is a displacement node, and vice-versa.

### 20.42 (a)

Here, we are thinking of displacement nodes and antinodes and not pressure (anti)nodes. The material at the center of the star cannot move so it must correspond to a displacement node. The surface is a pressure node and hence a displacement antinode.

**20.42 (b)**

If we picture a tube with a node at one end (source) and an antinode at the other (open) end, then it is easy to see that (in the fundamental mode) the length of this tube corresponds to one-quarter of a wavelength.

In our case the length of the tube corresponds to the average radius of the star i.e. $L = R$.

$$\begin{aligned} R &= \frac{\lambda}{4} \\ \Rightarrow \nu &= \frac{v}{4R} \text{ using } v = \nu\lambda \\ \frac{1}{T} &= \frac{v}{4R} \\ T &= \frac{4R}{v} \end{aligned}$$

**20.42 (c)**

$$\begin{aligned} v &= \sqrt{\frac{B}{\rho}} \\ v &= \sqrt{\frac{1.33 \times 10^{22}}{1 \times 10^{10}}} \\ v &= 1.153 \times 10^6 \ m/s \end{aligned}$$

$$\begin{aligned} R &= (.009)R_0 \qquad \text{Find a value for } R_0 \text{ in Appendix C} \\ R &= (.009)(6.955 \times 10^8) \\ R &= 6.26 \times 10^6 \end{aligned}$$

$$\begin{aligned} T &= \frac{4R}{v} \\ T &= \frac{4(6.26 \times 10^6)}{1.153 \times 10^6} \\ T &= 21.71 \ s \end{aligned}$$

## HRK 20.53

**20.53(a)**

You might have seen elsewhere that the number of ways of picking two items from a group of 5 different ones is

$$\text{“ Five choose two ”: } \binom{5}{2} = 10$$

You can figure this question out, however, without knowing anything about combinatorics.

Every pair of forks has a beat frequency $\nu_{beat} = |\nu_i - \nu_j|$. We can arrange it so that every such pair has a beat frequency that is different to that of any other pair. For example, consider $\{\nu_1 = 2, \nu_2 = 4, \nu_3 = 8, \nu_4 = 16, \nu_5 = 32\}$, then the ten distinct beat frequencies are

$$\begin{array}{ccccc} \nu_{1,2}^{beat} = 2, & \nu_{1,3}^{beat} = 6, & \nu_{1,4}^{beat} = 14, & \nu_{1,5}^{beat} = 30, & \nu_{2,3}^{beat} = 4, \\ \nu_{2,4}^{beat} = 12, & \nu_{2,5}^{beat} = 28, & \nu_{3,4}^{beat} = 8, & \nu_{3,5}^{beat} = 24, & \nu_{4,5}^{beat} = 16 \end{array}$$

where the notation I have used is

$$\nu_{i,j}^{beat} = |\nu_i - \nu_j|.$$

Obviously $\nu_{i,j}^{beat} = \nu_{j,i}^{beat}$ so I don't bother listing terms like $\nu_{4,2}^{beat} = \nu_{2,4}^{beat}$.

**20.53(b)**

In trying to find a worst case scenario we could arrange it so that the a number of the pairs have the same beat frequency. Since the forks are all distinct the worst we can come up with is something like $\{\nu_1 = 100, \nu_2 = 200, \nu_3 = 300, \nu_4 = 400, \nu_5 = 500\}$ where

$$|\nu_1 - \nu_2| = |\nu_2 - \nu_3| = |\nu_3 - \nu_4| = |\nu_4 - \nu_5|$$

and also

$$|\nu_1 - \nu_3| = |\nu_2 - \nu_4| = |\nu_3 - \nu_5|$$

and finally

$$|\nu_1 - \nu_4| = |\nu_2 - \nu_5|$$

If you select one representative pairing from each of the above groups and add in the final possible beat frequency $|\nu_1 - \nu_5|$ then we have a total of 4 distinct beat frequencies

$$\{|\nu_1 - \nu_2|, |\nu_1 - \nu_3|, |\nu_1 - \nu_4|, |\nu_1 - \nu_5|\}$$

**Note:** We know that 4 is the lowest achievable since all 5 forks are distinct, and straight away that implies that none of the 4 beat frequencies listed above $\{|\nu_1 - \nu_2|, |\nu_1 - \nu_3|, |\nu_1 - \nu_4|, |\nu_1 - \nu_5|\}$ can be same.

## HRK 20.70

**20.70(a)**

The source is moving towards the observer. The observer is moving towards the source. They are both moving with respect to the medium (water). Convert speeds to meters/second first.

$$\begin{aligned}
20.2\ km/h &= \tfrac{2.02\times10^4\ m}{3600\ s} &= 5.611 m/s \\
94.6\ km/h &= \tfrac{9.46\times10^4\ m}{3600\ s} &= 26.277 m/s \\
5470\ km/h &= \tfrac{5.47\times10^6\ m}{3600\ s} &= 1519.44 m/s
\end{aligned}$$

Here sub 1 is the source and sub 2 is the observer. The frequency $\nu$ is 1030 Hz.

$$\begin{aligned}
\nu' &= \nu \frac{v + v_O}{v - v_S} \\
\nu' &= \nu \frac{1519.44 + 26.277}{1519.44 - 5.611} \\
\nu' &= \nu(1.02106) \\
\nu' &= (1030)(1.02106) \\
\nu' &= 1051.7\ Hz
\end{aligned}$$

**20.70(b)**

Here, we make sub 2 the source (since the waves are reflecting from sub 2) and sub 1 the observer. The frequency $\nu$ of the reflected waves is 1051.7 Hz, the frequency of the wave arriving at sub 2 (which we worked out in part (a)).

$$\begin{aligned}\nu' &= \nu \, \frac{v+v_O}{v-v_S} \\ \nu' &= \nu \, \frac{1519.44+5.611}{1519.44-26.277} \\ \nu' &= \nu(1.02136) \\ \nu' &= (1051.7)(1.02136) \\ \nu' &= 1074.16 \; Hz\end{aligned}$$

# Taylor Expansions:

## 1

We will construct the first few terms of the following series

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!}\dots$$

| $f, f', f''$ | Value at $x=0$ | Resulting Term |
|---|---|---|
| $f = \frac{1}{1+x}$ | $f(0) = 1$ | 1 |
| $f' = \frac{-1}{(1+x)^2}$ | $f'(0) = -1$ | $-x$ |
| $f'' = \frac{2}{(1+x)^3}$ | $f''(0) = 2$ | $x^2$ |
| $f''' = \frac{-6}{(1+x)^4}$ | $f'''(0) = -6$ | $-x^3$ |
| $f'''' = \frac{24}{(1+x)^5}$ | $f''''(0) = 24$ | $x^4$ |

Overall we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \dots$$

You could apply the ratio test to show that the series converges if $|x| < 1$.

## 2

$$\begin{aligned}f(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!}\dots \\ f(\theta) &= f(0) + f'(0)\theta + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + f''''(0)\frac{\theta^4}{4!}\dots\end{aligned}$$

From the 4th derivative onwards, you can see the first two columns are going to keep repeating, so that all even powers of $\theta$ in $f(0) + f'(0)\theta + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + f''''(0)\frac{\theta^4}{4!}\dots$ will be multiplied by zero. All the odd powers of $\theta$ will have sign $\pm 1$ (alternating).

$$\begin{aligned}\sin x &= x + (-1)\frac{x^3}{3!} + (1)\frac{x^5}{5!} + (-1)\frac{x^7}{7!}\dots \\ \sin\theta &= \theta + (-1)\frac{\theta^3}{3!} + (1)\frac{\theta^5}{5!} + (-1)\frac{\theta^7}{7!}\dots\end{aligned}$$

Table 1: Table for $f = \sin\theta$

| $f, f', f''$ | Value at $\theta = 0$ | Resulting Term |
|---|---|---|
| $f = \sin\theta$ | $f(0) = 0$ | $0$ |
| $f' = \cos\theta$ | $f'(0) = 1$ | $\theta$ |
| $f'' = -\sin\theta$ | $f''(0) = 0$ | $0$ |
| $f''' = -\cos\theta$ | $f'''(0) = -1$ | $\frac{-\theta^3}{3!}$ |
| $f'''' = \sin\theta$ | $f''''(0) = 0$ | $0$ |

## 3

Table 2: Table for $f = \cos\theta$

| $f, f', f''$ | Value at $\theta = 0$ | Resulting Term |
|---|---|---|
| $f = \cos\theta$ | $f(0) = 1$ | $1$ |
| $f' = -\sin\theta$ | $f'(0) = 0$ | $0$ |
| $f'' = -\cos\theta$ | $f''(0) = -1$ | $\frac{-\theta^2}{2!}$ |
| $f''' = \sin\theta$ | $f'''(0) = 0$ | $0$ |
| $f'''' = \cos\theta$ | $f''''(0) = 1$ | $\frac{\theta^4}{4!}$ |

From the 4th derivative onwards, you can see the first two columns are going to keep repeating, so that all odd powers of $\theta$ in $f(0) + f'(0)\theta + f''(0)\frac{\theta^2}{2!} + f'''(0)\frac{\theta^3}{3!} + f''''(0)\frac{\theta^4}{4!} \ldots$ will be multiplied by zero. All the even powers of $\theta$ will have sign $\pm 1$ (alternating).

$$\begin{aligned} \cos x &= 1 + (-1)\frac{x^2}{2!} + (1)\frac{x^4}{4!} + (-1)\frac{x^6}{6!} \ldots \\ \cos\theta &= 1 + (-1)\frac{\theta^2}{2!} + (1)\frac{\theta^4}{4!} + (-1)\frac{\theta^6}{6!} \ldots \end{aligned}$$

## 4

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \ldots$$

Table 3: Table for $f = e^x$

| $f, f', f''$ | Value at $\theta = 0$ | Resulting Term |
|---|---|---|
| $f = e^x$ | $f(0) = 1$ | 1 |
| $f' = e^x$ | $f'(0) = 1$ | $\frac{x}{1!}$ |
| $f'' = e^x$ | $f''(0) = 1$ | $\frac{x^2}{2!}$ |
| $f''' = e^x$ | $f'''(0) = 1$ | $\frac{x^3}{3!}$ |
| $f'''' = e^x$ | $f''''(0) = 1$ | $\frac{x^5}{5!}$ |

Now, setting $x = i\theta$, we get

$$e^{(i\theta)} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} \dots$$

Let us combine the series for $\cos\theta$ and $i\sin\theta$ into one series:

$$\begin{aligned}
\cos\theta &= 1 + (-1)\frac{\theta^2}{2!} + (1)\frac{\theta^4}{4!} + (-1)\frac{\theta^6}{6!} \dots \\
+i\sin\theta &= i\theta + (-i)\frac{\theta^3}{3!} + (i)\frac{\theta^5}{5!} + (-i)\frac{\theta^7}{7!} \dots \\
\Rightarrow \cos\theta + i\sin\theta &= 1 + i\theta + (-1)\frac{\theta^2}{2!} + (-i)\frac{\theta^3}{3!} + (1)\frac{\theta^4}{4!} + (i)\frac{\theta^5}{5!} + (-1)\frac{\theta^6}{6!} + (-i)\frac{\theta^7}{7!} \dots
\end{aligned}$$

The last step is to write all the $\pm 1, \pm i$ coefficients as powers of $i$ – then we immediately see Euler's Theorem:

$$\cos\theta + i\sin\theta = 1 + i\theta + (i^2)\frac{\theta^2}{2!} + (i^3)\frac{\theta^3}{3!} + (i^4)\frac{\theta^4}{4!} + (i^5)\frac{\theta^5}{5!} + (i^6)\frac{\theta^6}{6!} + (i^7)\frac{\theta^7}{7!} \dots = e^{(i\theta)}$$