HRK 21.2

From the discussion of length contraction and time dilation we would expect the tube to appear shorter (in the direction of motion) when we are moving at high speed with respect to it. The proper length L_0 is the length of the tube as measured in the reference frame of the tube (i.e the laboratory frame). If we imagine ourselves as observers, moving along with the electron, then we see the tube (and, in general, the whole lab) rush by at speed $0.999987c$. The relevant transformation is

$$L = \frac{L_0}{\gamma} \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{0.999987c^2}{c^2}}} = 196.1$$

Plugging in the numbers, we find the observed tube length L when observed by someone traveling with the electron (i.e. observed by someone in frame S') is

$$L = \frac{2.86}{196.1} = 1.458 \times 10^{-2}\ m$$

HRK 21.4

The proper time Δt_0 of the muons' mean lifetime is $2.20\mu s$. If we were travelling along with the muon, this is how long we would see it last. By the phenomenon of time dilation, we expect muons traveling at high speeds relative to us to have a longer observed lifetime Δt (in this case $\Delta t = 16\mu s$). The relationship between the two periods of time is given by

$$\Delta t = \gamma \Delta t_0 = \frac{\Delta t_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

We know Δt_0 and Δt so let us rearrange in order to solve for u, the speed of the muons in the laboratory (i.e. Earth) reference frame

$$\begin{aligned}
\frac{\Delta t}{\gamma} &= \Delta t_0 \\
\frac{1}{\gamma} &= \frac{\Delta t_0}{\Delta t} \\
\frac{1}{\gamma^2} &= \frac{\Delta t_0}{\Delta t} \\
(1 - \frac{u^2}{c^2}) &= \frac{\Delta t_0^2}{\Delta t^2} \\
\frac{u^2}{c^2} &= 1 - \frac{\Delta t_0^2}{\Delta t^2} \\
u &= c\sqrt{1 - \frac{\Delta t_0^2}{\Delta t^2}} \\
u &= c\sqrt{1 - \frac{(2.2 \times 10^{-6})^2}{(16 \times 10^{-6})^2}} \\
u &= .990502\ c
\end{aligned}$$

HRK 21.5

Method 1 (More Direct)

For an observer in the lab frame, they would see see the particle travel a distance $1.05 \times 10^{-3}\ m$ while moving at a speed of $v = 0.992\ c$. The observed lifetime of the particle in the lab frame is then

$$\Delta t_{LAB} = \frac{1.05 \times 10^{-3}}{0.992\ c} = \frac{1.05 \times 10^{-3}}{(0.992)(3 \times 10^8)} = 3.528 \times 10^{-12}\ s$$

We know from our discussions of time dilation that an observer traveling along with particle would see the proper time Δt_0 for the lifetime of the particle, and we know that the proper time is related to the time in lab frame via

$$\Delta t_{LAB} = \gamma \Delta t_0 \qquad \text{where } \gamma = \frac{1}{\sqrt{1-0.992^2}}$$

and therefore

$$\begin{aligned}
\Delta t_0 &= \frac{\Delta t_{LAB}}{\gamma} \\
\Delta t_0 &= (3.528 \times 10^{-12})\sqrt{1-0.992^2} \\
\Delta t_0 &= 4.454 \times 10^{-13}\ s
\end{aligned}$$

Method 2 (Slightly Circuitous)

The key idea this time is that an observer traveling with particle would measure the distance between when the particle entered the detector, to when it stopped, as being some length ΔL which is shorter than the the observed length of travel ΔL_0 in the reference frame of the laboratory (here, a detector). This is the phenomenon of length contraction.

An observer traveling along with the particle would have seen the detector move by at a speed of $u = 0.992c$ and this observer would measure the distance traveled into the detector as

$$\begin{aligned}
\Delta L &= \frac{\Delta L_0}{\gamma} \qquad \text{where } \gamma = \frac{1}{\sqrt{1-0.992^2}} \\
\Delta L &= (1.05 \times 10^{-3})\sqrt{1-0.992^2} \\
\Delta L &= 1.3255 \times 10^{-4}\ m
\end{aligned}$$

For an observer traveling with the particle, since the detector was moving by at $u = 0.992c$ and since it traveled a distance of $1.3255 \times 10^{-4}\ m$, the amount of time the particle lasted must then be

$$\begin{aligned}
\text{time} &= \frac{\text{dist}}{\text{speed}} \\
\Delta t_0 &= \frac{1.3255 \times 10^{-4}\ m}{(0.992)(3 \times 10^8)} \\
\Delta t_0 &= 4.454 \times 10^{-13}\ s
\end{aligned}$$

HRK 21.14

There is nothing too tricky going on here. This is just like the normal set up in which we imagine tow coordinate systems S and S', with S' moving in the positive x direction (which is also the x' direction) at constant speed u with respect to S. Note that the Lorentz factor γ for this problem has a value $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.95)^2}} = 3.20256$.

Answering this question requires a straightforward application of the Lorentz transformation equations

$$\begin{aligned}
x' &= \gamma(x - ut) \\
y' &= y \\
z' &= z \\
t' &= \gamma\left(t - \frac{ux}{c^2}\right)
\end{aligned}$$

We know the y and z coordinates are the same in both frames S and S' so concentrate on x' and t' using the given information

$$\begin{aligned}
x &= 100 \times 10^3\ m \\
t &= 200 \times 10^{-6}\ s \\
u &= 0.95\ c = 2.85 \times 10^8\ m/s
\end{aligned}$$

$$
\begin{aligned}
x' &= \gamma(x-ut) = (3.2025)\left[(100\times 10^3) - (2.85\times 10^8)(200\times 10^{-6})\right] = 1.377\times 10^5\ m \\
t' &= \gamma\left(t-\frac{ux}{c^2}\right) = (3.2025)\left[(200\times 10^{-6}) - \frac{(0.95)(100\times 10^3)}{(3\times 10^8)}\right] = -3.736\times 10^{-4}\ s
\end{aligned}
$$

HRK 21.17

It should be obvious from the wording of the question that what we are interested in here is the intervals in space and time between two events, in two different coordinate systems. We let the experimenter occupy reference frame S, and so the second observer is in the reference frame S'. We can solve this problem by either using the regular Lorentz transformation equations, or by using the interval version of these transformation equations. The choice of which version to use is purely one of convenience – so we will do it both ways here.

Method 1: Regular Lorentz Transformation equations

21.17 (a)

The relevant equations are

$$
\begin{aligned}
x' &= \gamma(x-ut) \\
t' &= \gamma\left(t-\frac{ux}{c^2}\right)
\end{aligned}
$$

We will put subscripts on the x and t to denote which color light we are talking about. Let us say that that the blue light flashes at $t_b = 0$ and at location $x_b = 0$. Since the red and blue flash simultaneously (to the experimenter) we have $t_r = 0$ too, and we also know that $x_r = 30.4\times 10^3\ m$. Before we start plugging in values, note that the Lorentz factor is $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.247)^2}} = 1.032$.

$$
\begin{aligned}
x'_b &= \gamma(x_b - ut_b) = (1.032)\left[0 - u0\right] = 0 \\
x'_r &= \gamma(x_r - ut_r) = (1.032)\left[30.4\times 10^3 - u0\right] = 3.1372\times 10^4\ m \\
t'_b &= \gamma\left(t_b - \frac{ux_b}{c^2}\right) = (1.032)\left[0 - \frac{u0}{c^2}\right] = 0 \\
t'_r &= \gamma\left(t_r - \frac{ux_r}{c^2}\right) = (1.032)\left[0 - \frac{(0.247)(30.4\times 10^3)}{c}\right] = -2.583\times 10^{-5}\ s
\end{aligned}
$$

21.17 (b)

Since $t'_r < t'_b$ we know that the red light must flash first according to the observer in the frame S'.

Method 2: Interval Lorentz Transformation equations

21.17 (a)

The fact that the bulbs flash simultaneously in S tell us that time interval between the events in this frame is $t_r - t_b = \Delta t = 0$ where the r and b subscripts correspond to the colors of the lights. The spatial separation between these two flashes, in S, is $\Delta x = x_r - x_b = 30.4\times 10^3\ m$.

The interval version of the Lorentz transformation equations is given by

$$
\begin{aligned}
\Delta x' &= \gamma(\Delta x - u\Delta t) \\
y' &= y \\
z' &= z \\
\Delta t' &= \gamma\left(\Delta t - \frac{u\Delta x}{c^2}\right)
\end{aligned}
$$

Since the observer's frame S' is moving at $u = 0.247c$ with respect to S (in the positive x direction) then the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.247)^2}} = 1.032$.

Find the spatial and temporal intervals ($\Delta x'$ and $\Delta t'$ respectively) between the two flashes in the S' frame by plugging in the above values:

$$\begin{aligned}\Delta x' &= (1.032)\left[(30.4\times 10^3)-(0.247c)(0)\right] = 3.1372\times 10^4\ m\\ y' &= y\\ z' &= z\\ \Delta t' &= (1.032)\left[0-\frac{(0.247)(30.4\times 10^3)}{(3\times 10^8)}\right] = -2.583\times 10^{-5}\ s\end{aligned}$$

We defined $\Delta x = x_r - x_b$ and $\Delta t = t_r - t_b$ so the relevant expressions for the S' frame are

$$\begin{aligned}\Delta x' &= x'_r - x'_b\\ \Delta t' &= t'_r - t'_b\end{aligned}$$

21.17 (b)

Since $\Delta t' < 0$ that means that $t'_r < t'_b$ and consequently the red light flashes first.

HRK 21.18

We want to invert the relationships

$$\begin{aligned}x' &= \gamma(x-ut)\\ y' &= y\\ z' &= z\\ t' &= \gamma(t-\frac{ux}{c^2})\end{aligned}$$

The job is already done for y' and z'. For x' and t' there are a couple of ways to arrive at the desired expressions.

Method 1: shorter

$$\begin{aligned}x' &= \gamma(x-ut)\\ \Rightarrow x' &= \gamma x - \gamma ut\end{aligned}$$

and also

$$\begin{aligned}t' &= \gamma(t-\frac{ux}{c^2})\\ t' &= \gamma t - \frac{\gamma ux}{c^2}\\ \Rightarrow ut' &= u\gamma t - \frac{\gamma u^2 x}{c^2}.\end{aligned}$$

Now add these two expressions

$$\begin{aligned}x' &= \gamma x - \gamma ut\\ +ut' &= u\gamma t - \frac{\gamma u^2 x}{c^2}\\ \hline \Rightarrow x' + ut' &= \gamma x - \frac{\gamma u^2 x}{c^2} + u\gamma t - \gamma ut\\ x' + ut' &= \gamma x\left(1-\frac{u^2}{c^2}\right)\\ x' + ut' &= \frac{x}{\gamma} \qquad \text{using } \gamma^{-2} = \left(1-\frac{u^2}{c^2}\right)\\ \gamma(x'+ut') &= x \qquad \text{as required.}\end{aligned}$$

Now we do a similar thing to get the expression for t.

$$\begin{aligned} x' &= \gamma(x - ut) \\ x' &= \gamma x - \gamma ut \\ \Rightarrow \frac{ux'}{c^2} &= \frac{u\gamma x}{c^2} - \frac{\gamma u^2 t}{c^2} \end{aligned}$$

and also

$$\begin{aligned} t' &= \gamma(t - \frac{ux}{c^2}) \\ \Rightarrow t' &= \gamma t - \frac{\gamma ux}{c^2} \end{aligned}$$

Now add these two expressions

$$\begin{aligned} \frac{ux'}{c^2} &= \frac{u\gamma x}{c^2} - \frac{\gamma u^2 t}{c^2} \\ +t' &= \gamma t - \frac{\gamma ux}{c^2} \\ \hline \Rightarrow \frac{ux'}{c^2} + t' &= \frac{u\gamma x}{c^2} - \frac{\gamma ux}{c^2} + \gamma t - \frac{\gamma u^2 t}{c^2} \end{aligned}$$

$$\begin{aligned} \frac{ux'}{c^2} + t' &= \gamma t - \frac{\gamma u^2 t}{c^2} \\ \frac{ux'}{c^2} + t' &= \gamma t \left(1 - \frac{u^2}{c^2}\right) \\ \frac{ux'}{c^2} + t' &= \frac{t}{\gamma} \qquad \text{using } \gamma^{-2} = \left(1 - \frac{u^2}{c^2}\right) \\ \gamma\left(\frac{ux'}{c^2} + t'\right) &= t \qquad \text{as required.} \end{aligned}$$

Method 2: Longer

We will just multiply out and re-arrange

$$\begin{aligned} x' &= \gamma(x - ut) \\ x' + \gamma ut &= \gamma x \\ \Rightarrow x &= \frac{x'}{\gamma} + ut \end{aligned}$$

We want x in terms of γ, x' and t' so we must replace t in the last line. We will come back to this in a minute. For now, let us rearrange the expression for t'

$$\begin{aligned} t' &= \gamma(t - \frac{ux}{c^2}) \\ \gamma t &= t' + \gamma\frac{ux}{c^2} \\ \Rightarrow t &= \frac{t'}{\gamma} + \frac{ux}{c^2} \end{aligned}$$

Plug $t = \frac{t'}{\gamma} - \frac{ux}{c^2}$ into $x = \frac{x'}{\gamma} + ut$ and do some tidying

$$\begin{aligned}
x &= \frac{x'}{\gamma} + u\left(\frac{t'}{\gamma} + \frac{ux}{c^2}\right) \\
x - \frac{u^2 x}{c^2} &= \frac{x'}{\gamma} + \frac{ut'}{\gamma} \\
x\left(1 - \frac{u^2}{c^2}\right) &= \frac{x' + ut'}{\gamma} \\
x\gamma^{-2} &= \frac{x' + ut'}{\gamma} \qquad \text{using } \gamma^{-2} = \left(1 - \frac{u^2}{c^2}\right) \\
\Rightarrow x &= \gamma(x' + ut')
\end{aligned}$$

Similarly, to get t in terms γ, x' and t' we substitute $x = \gamma(x' + ut')$ into $t = \frac{t'}{\gamma} + \frac{ux}{c^2}$

$$\begin{aligned}
t &= \frac{t'}{\gamma} + \frac{u\gamma(x' + ut')}{c^2} \\
t &= \frac{t'}{\gamma} + \frac{u^2\gamma t'}{c^2} + \frac{u\gamma x'}{c^2} \\
t &= \gamma\left(\frac{t'}{\gamma^2} + \frac{u^2 t'}{c^2} + \frac{ux'}{c^2}\right) \\
t &= \gamma\left(t'\left(\frac{1}{\gamma^2} + \frac{u^2}{c^2}\right) + \frac{ux'}{c^2}\right) \\
t &= \gamma\left(t'\left(1 - \frac{u^2}{c^2} + \frac{u^2}{c^2}\right) + \frac{ux'}{c^2}\right) \qquad \text{using } \gamma^{-2} = \left(1 - \frac{u^2}{c^2}\right) \\
\Rightarrow t &= \gamma\left(t' + \frac{ux'}{c^2}\right)
\end{aligned}$$

HRK 21.21

21.21(a)

If galaxy A is receding from us with speed $0.347c$ (in our reference frame) then we are receding from it with the same speed (in its reference frame).

21.21(b)

Let's stick to the usual convention of calling the lab frame S (here lab=earth). If galaxy A is moving in our positive x direction, with speed $0.347c$, and we let S' be the reference frame associated with galaxy A, then $\vec{u} = 0.347\hat{x}$. Since galaxy B is receding from us in the opposite direction we know that we would observe B to have velocity $\vec{v} = -0.347c$. What we are trying to find is the velocity $\vec{v'}$ that an observer in S' would find for galaxy B. The relevant expression is

The relevant Lorentz velocity transformation is

$$\begin{aligned}
v'_x &= \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \qquad \text{where } v_x = -0.347c \text{ and } u = 0.347c \\
v_x &= \frac{-0.347c - 0.347c}{1 - \frac{-(0.347c)^2}{c^2}} \\
v_x &= -0.6194c
\end{aligned}$$

HRK 21.24

21.24(a)

Note that if we just used Gallilean relativity (and not special relativity) we would find that the electron is moving faster than c in the lab frame – which is nonsense.

Once we have agreed upon the definitions for the various ingredients S, S', $\vec{u}$, $\vec{v}$ and $\vec{v'}$ then the question will boil down to a straightforward application of one of the Lorentz velocity transformations.

S: the laboratory frame

S': the frame moving along with the **nucleus**.

$\vec{u}$: the velocity of the S' frame relative to S. This is the velocity of the nucleus relative to the lab frame i.e. $\vec{u} = 0.24c\ \hat{x}$.

$\vec{v}$: The velocity of the **electron** in the **lab frame**. This is what we are trying to find. The components are $\vec{v} = (v_x, v_y, v_z)$

$\vec{v'}$: The velocity of the **electron** in the **nucleus' frame**, S'. We are given that this is $\vec{v'} = 0.78c\ \hat{x}$. The individual components are $\vec{v'} = (v'_x = 0.78c, v'_y = 0, v'_z = 0)$

The version of Lorentz's velocity transformation that is most helpful to us is

$$\begin{aligned} v_x &= \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \\ v_y &= \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \\ v_z &= \frac{v'_z}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \end{aligned}$$

and so plugging in the values we get

$$\begin{aligned} v_x &= \frac{0.78c + (0.24c)}{1 + \frac{(0.24c)(0.78c)}{c^2}} \\ v_y &= \frac{0}{\gamma\left(1 + \frac{(0.24c)(0.78c)}{c^2}\right)} \\ v_z &= \frac{0}{\gamma\left(1 + \frac{(0.24c)(0.78c)}{c^2}\right)} \end{aligned}$$

The Lorentz factor is $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.24)^2}} = 1.03011$ and the quantity $\left(1 + \frac{(0.24c)(0.78c)}{c^2}\right)$ has a value 1.1872

$$\begin{aligned} v_x &= \frac{1.02c}{1.1872} = 0.8592c \\ v_y &= \frac{0}{(1.03011)(1.1872)} = 0 \\ v_z &= \frac{0}{(1.03011)(1.1872)} = 0 \end{aligned}$$

The magnitude of $\vec{v}$ is

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = 0.8592c$$

21.24(b)

The set up here is the same as part (a) except now the emitted electron moves along the y' axis and not the x' axis. List out the ingredients again, noting that $\vec{v'}$ must be modified from part (a):

S: the laboratory frame

S': the frame moving along with the **nucleus**.

$\vec{u}$: the velocity of the S' frame relative to S. This is the velocity of the nucleus relative to the lab frame i.e. $\vec{u} = 0.24c\ \hat{x}$.

$\vec{v}$: The velocity of the **electron** in the **lab frame**. This is what we are trying to find. The components are $\vec{v} = (v_x, v_y, v_z)$

$\vec{v'}$: The velocity of the **electron** in the **nucleus' frame**, S'. We are given that this is $\vec{v'} = 0.78c\ \hat{y}'$. The individual components are $\vec{v'} = (v'_x = 0, v'_y = 0.78c, v'_z = 0)$

$$\begin{aligned} v_x &= \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \\ v_y &= \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \\ v_z &= \frac{v'_z}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \end{aligned}$$

$$\begin{aligned} v_x &= \frac{0 + (0.24c)}{1 + \frac{(0.24c)(0)}{c^2}} = 0.24c \\ v_y &= \frac{(0.78c)}{\gamma\left(1 + \frac{(0.24c)(0)}{c^2}\right)} = \frac{0.78}{1.03011}c = 0.7572c \\ v_z &= \frac{0}{\gamma\left(1 + \frac{(0.24c)(0)}{c^2}\right)} = 0 \end{aligned}$$

The magnitude of $\vec{v}$ is

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(0.24)^2 + (0.7572)^2 + 0} = 0.7943c$$

The direction of motion is in the positive quadrant of the $x - y$ plane and the velocity vector makes an angle of $\arctan(\frac{0.757}{0.24}) = 72.4°$ with the $x-$axis.

21.24(c)

This time we are given the electron's velocity $\vec{v}$, as measured by someone in S, and we have to figure out what someone in S' would measure the electron's velocity as (i.e. we have to figure out $\vec{v'}$).

S: the laboratory frame

S': the frame moving along with the **neutron**.

$\vec{u}$: the velocity of the S' frame relative to S. This is the velocity of the neutron relative to the lab frame i.e. $\vec{u} = 0.24c\ \hat{x}$.

$\vec{v}$: The velocity of the **electron** in the **lab frame**.Now we are given this – its velocity is $\vec{v} = 0.78c\ \hat{y}$. The components are $\vec{v} = (v_x = 0, v_y = 0.78c, v_z = 0)$

$\vec{v'}$: The velocity of the **electron** in the **neutron's frame**, S'. This is what we are trying to figure out. The components are $\vec{v'} = (v'_x, v'_y, v'_z)$.

We will use the other version of Lorentz's velocity transformation equation, since that is more convenient.

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}}$$
$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)}$$
$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{uv_x}{c^2}\right)}$$

Plug in numbers

$$v'_x = \frac{0 - (0.24c)}{1 - \frac{(0.24c)(0)}{c^2}} = -0.24c$$
$$v'_y = \frac{0.78c}{\gamma\left(1 - \frac{(0.24c)(0)}{c^2}\right)} = \frac{0.78}{(1.03011)}c = 0.7572c$$
$$v'_z = \frac{0}{\gamma\left(1 - \frac{(0.24c)(0)}{c^2}\right)} = 0$$

The magnitude of $\vec{v'}$ is

$$\sqrt{(v'_x)^2 + (v'_y)^2 + (v'_z)^2} = \sqrt{(0.24)^2 + (0.7572)^2 + 0} = 0.7943c$$

The direction of motion is in the quadrant of the $x' - y'$ plane with $x' < 0$ and $y' > 0$, and the velocity vector makes an angle of $\arctan(\frac{0.757}{0.24}) = 72.41°$ with the $x'-$axis.

Note: Compare this with the angle we found in part (b). In part (b), a velocity that was purely y' directed in the S' frame, was observed in the S frame to make an angle of $+72.41°$ with the x axis. In part (c), a velocity that was purely y directed in the S frame, was observed in the S' frame to make an angle of $-72.41°$ with the x' axis. Thinking about the relative motion between frame S and frame S' you can convince yourself that such a relationship must always hold.

Ambiguous wording in the question

Depending on how you read the question, you could double the amount of work involved if, for each of $(a), (b)$ and (c), you let the electron's velocity be directed in the positive and negative directions of the axis they refer to i.e. $\pm\hat{x'}, \pm\hat{y'}, \pm\hat{y}$ respectively. Anyone who interpreted the question this way will get extra credit for each correct answer.

For example

(a) again

S: the laboratory frame

S': the frame moving along with the **nucleus**.

$\vec{u}$: the velocity of the S' frame relative to S. This is the velocity of the nucleus relative to the lab frame i.e. $\vec{u} = 0.24c\ \hat{x}$.

$\vec{v}$: The velocity of the **electron** in the **lab frame**. This is what we are trying to find. The components are $\vec{v} = (v_x, v_y, v_z)$

$\vec{v'}$: The velocity of the **electron** in the **nucleus' frame**, S'. We now change this to $\vec{v'} = -0.78c\ \hat{x}$. The individual components are $\vec{v'} = (v'_x = -0.78c, v'_y = 0, v'_z = 0)$

and so

$$\begin{aligned} v_x &= \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} \\ v_y &= \frac{v'_y}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \\ v_z &= \frac{v'_z}{\gamma\left(1 + \frac{uv'_x}{c^2}\right)} \end{aligned}$$

and so plugging in the values we get

$$\begin{aligned} v_x &= \frac{-0.78c + (0.24c)}{1 + \frac{(0.24c)(-0.78c)}{c^2}} = -0.6644c \\ v_y &= \frac{0}{\gamma\left(1 + \frac{(0.24c)(-0.78c)}{c^2}\right)} = 0 \\ v_z &= \frac{0}{\gamma\left(1 + \frac{(0.24c)(-0.78c)}{c^2}\right)} = 0 \end{aligned}$$

HRK 21.31

Obviously this question will involve using the interval version of Lorentz's transformation equation in some way.

21.31(a)

The information given to us is that

The spatial separation between events "red light flashes" and "blue light flashes" $\Delta x = x_r - x_b = 730\ m$

The temporal separation between events "red light flashes" and "blue light flashes" $\Delta t = t_r - t_b = -4.96\mu s$ **(Note the minus sign)**

We want to find a frame S' in which the spatial separation between events "red light flashes" and "blue light flashes" is $\Delta x' = x'_r - x'_b = 0\ m$

There is no need to bring y and z coordinates into this problem. We will solve this in one spatial dimension $\hat{x} = \hat{x'}$.

The most useful interval version of Lorentz's transformation equations are

$$\Delta x' = \gamma(\Delta x - u\Delta t)$$
$$\Delta t' = \gamma(\Delta t - \frac{u\Delta x}{c^2})$$

Setting $\Delta x' = 0$ as required, we get

$$\begin{aligned} \Rightarrow \Delta x &= u\Delta t \\ \frac{\Delta x}{\Delta t} &= u \\ \frac{730}{-4.96 \times 10^{-6}} &= -1.4717 \times 10^8 = u \\ -0.4906c &= u \end{aligned}$$

This result tells us that frame S' should be moving with speed $0.4906c$ relative to the frame S in the negative $\hat{x} = \hat{x'}$ direction, in order for an observer in S' to see $\Delta x' = 0$.

21.31(b)

Given that we have now found u, let us use that to see what the temporal separation $\Delta t' = t'_r - t'_b$ is between the two events 'red light flashes" and "blue light flashes".

$$\begin{aligned}\Delta t' &= \gamma(\Delta t - \frac{u\Delta x}{c^2}) \\ \Delta t' &= \gamma(-4.96 \times 10^{-6} - \frac{(-0.4906c)(730)}{c^2})\end{aligned}$$

and, calculating the Lorentz factor to be $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.4906)^2}} = 1.1476$ we get

$$\begin{aligned}\Delta t' &= (1.1476)(-4.96 \times 10^{-6} - \frac{(-0.4906c)(730)}{c^2}) \\ \Delta t' &= -4.322 \times 10^{-6} s\end{aligned}$$

Since $\Delta t' = t'_r - t'_b < 0$ this means that $t'_r < t'_b$ and the observer in S' sees the red light flash first.

HRK 21.32

Same setup as the last question but now we must leave the temporal separation Δt in frame S as an unknown.

The spatial separation between events "red light flashes" and "blue light flashes" $\Delta x = x_r - x_b = 730\ m$

The temporal separation between events "red light flashes" and "blue light flashes" $\Delta t < 0$ **We retain the ordering red-then-blue – hence the minus sign**

We want to find a frame S' in which the spatial separation between events "red light flashes" and "blue light flashes" is $\Delta x' = x'_r - x'_b = 0\ m$

The most useful interval version of Lorentz's transformation equations are again

$$\Delta x' = \gamma(\Delta x - u\Delta t)$$
$$\Delta t' = \gamma(\Delta t - \frac{u\Delta x}{c^2})$$

Setting $\Delta x' = 0$ as required, we get

$$\begin{aligned}\Rightarrow \Delta x &= u\Delta t \\ \Delta x &= u(t_r - t_b) \\ \frac{\Delta x}{(t_r - t_b)} &= u\end{aligned}$$

Recall, we said that red flashes first so that $t_r < t_b$ and hence the quantity $(t_r - t_b)$ in the denominator is negative. That implies the velocity u, of the observer in the x direction, must be negative (since Δx is positive). Also, we want the magnitude of this quantity $(t_r - t_b)$ to be as small as possible. This necessitates making the magnitude of u as large as possible. Taking all of this into account we arrive at the result that setting $\vec{u} = -c\hat{x}$ makes the magnitude of $(t_r - t_b)$ as small as possible, whilst retaining the order red-then-blue.

$$\begin{aligned}\frac{730}{(t_r - t_b)} &= -c \\ \frac{730}{-c} &= (t_r - t_b) \\ -2.433 \times 10^{-6} s &= (t_r - t_b)\end{aligned}$$

So, in order for it to be possible to see events "red light flashes" and then "blue light flashes" as occurring at the same location in S', we must have the red light flash *at least* $2.433 \times 10^{-6} s$ before the blue light in the S frame.