HRK 22.2

The thermometric property X that we are using is voltage, of which temperature is a linear function (on this practical scale, not actual thermodynamic temperature) i.e.

$$T(X) = aX + b$$

We are given two reference measurements so we can identify the unknown a and b.

$$\begin{aligned} T(X = 0\ mV) &= a(0) + b = 0°C \qquad \Rightarrow b = 0 \\ T(X = 28\ mV) &= a(28\ mV) + 0 = 510°C \qquad \Rightarrow a = \frac{510°C}{28\ mV} \end{aligned}$$

Now we can find the corresponding temperature when the voltage reading is 10.2 mV

$$\begin{aligned} T(X) &= \frac{510°C}{28\ mV} X \\ T(10.2\ mV) &= \frac{510°C}{28\ mV} 10.2\ mV = 185.79°C \end{aligned}$$

HRK 22.9

We want to find an expression for ΔT as a function of time, given that

$$\frac{d\Delta T}{dt} = -A(\Delta T).$$

22.9(a)

First look at the units on both sides of the equation above

$$\begin{aligned} \frac{d\Delta T}{dt} &= -A(\Delta T) \\ \frac{[K]}{[s]} &= -[A][K] \end{aligned}$$

Therefore, the units of A must be inverse time $\frac{1}{s}$.

22.9(b)

The units tell us that A must be the inverse of some kind of a so-called "time constant" which is a property of the system that is cooling. As we will see, A^{-1} is the time required for ΔT to relax to $\frac{\Delta T}{e}$. Physically you would expect the rate of cooling (or heating) to be related to quantities like surface area, mass, specific heat etc. You would therefore expect some combination of these kinds of these properties to have overall units of $\frac{1}{s}$.

22.9(c)

$$
\begin{aligned}
\frac{d\Delta T}{dt} &= -A(\Delta T) \\
\frac{d\Delta T}{(\Delta T)} &= -A\ dt \\
\int_{\Delta T_0}^{\Delta T(t)} \frac{d\Delta T}{(\Delta T)} &= -\int_0^t A\ dt \qquad \text{using } \Delta T(t=0) = \Delta T_0 \\
\ln(\Delta T)\Big]_{\Delta T_0}^{\Delta T(t)} &= -At \\
\ln(\Delta T(t)) - \ln(\Delta T_0) &= -At \\
\ln\left(\frac{\Delta T(t)}{\Delta T_0}\right) &= -At \\
\frac{\Delta T(t)}{\Delta T_0} &= e^{-At} \\
\Delta T(t) &= \Delta T_0 e^{-At}
\end{aligned}
$$

Note that if you evaluate $\Delta T(t)$ at $t = A^{-1}$ we get

$$
\begin{aligned}
\Delta T(A^{-1}) &= \Delta T_0 e^{-1} \\
\Delta T(A^{-1}) &= \frac{\Delta T_0}{e}
\end{aligned}
$$

HRK 22.13

Using figure 5 in your text book, you need to read off $\Delta T = T_N - T_{He}$ at a triple point value of $p_{tr} = 100\ cm\ Hg$.

$$\Delta T = 0.2K \qquad \text{at } p_{tr} = 100\ cm\ Hg$$

$$
\begin{aligned}
\Delta T &= (273.16)\frac{\Delta p}{p_{tr}} \\
0.2 &= (273.16)\frac{p_N - p_{He}}{100} \\
\frac{0.2}{273.16}(100) &= p_N - p_{He} \\
.0732\ cm\ Hg &= p_N - p_{He}
\end{aligned}
$$

HRK 22.16

Method 1

If you recall from your discussion of expansion, every line in an isotropic material lengthens by a factor

$$\frac{\Delta L}{L} = \alpha \Delta T$$

for a given increase in temperature ΔT. Since the diameter d is just a particular line in this isotropic material we can say

$$\begin{aligned}
\frac{\Delta d}{d} &= \alpha\Delta T \\
\frac{d_f - d_i}{d_i} &= \alpha\Delta T \qquad (\textbf{f}\text{inal, } \textbf{i}\text{nitial}) \\
d_f &= d_i + d_i\alpha\Delta T \\
d_f &= d_i(1+\alpha\Delta T) \\
d_f &= 2.725(1+(23\times 10^{-6})(138)) \\
d_f &= 2.73365\ cm
\end{aligned}$$

Note that the hole gets bigger, not smaller as people sometimes think it should. Remember all the atomic bond lengths are increasing with temperature, so the circumference must increase.

Method 2

$$\begin{aligned}
\Delta A &= A_f - A_i \\
\Rightarrow A_f &= A_i + \Delta A
\end{aligned}$$

Using the formula for change in area per change in temperature

$$\Delta A = 2\alpha A_i\Delta T$$

we see that the final area A_f is related to the initial area A_i via

$$A_f = A_i(1+2\alpha\Delta T)$$

Let us re-write the areas in terms of the radii as this is more useful to us.

$$\pi r_f^2 = \pi r_i^2(1+2\alpha\Delta T)$$

Canceling π and taking the square root of both sides

$$r_f = r_i\sqrt{(1+2\alpha\Delta T)}$$

Finally, multiply both sides by two to gives us the diameter instead of the radius

$$\begin{aligned}
d_f &= d_i\sqrt{(1+2\alpha\Delta T)} \\
d_f &= (2.725\ cm)\sqrt{1+2(23\times 10^{-6})(138)} \\
d_f &= 2.73364\ cm
\end{aligned}$$

HRK 22.19

22.19(a)

The surface area of the cube is just 6 times the area A of any particular face. We know that the area will change by

$$\begin{aligned}
\Delta A &= 2\alpha A\Delta T \\
\Delta\text{ Surface Area} &= (6)(2\alpha A\Delta T) \\
\Delta\text{ Surface Area} &= (6)(2)(19\times 10^{-6})(33.2^2)(55) \\
\Delta\text{ Surface Area} &= 13.82\ cm^2
\end{aligned}$$

This seems like quite a lot – is it? Check the fractional change in surface area

$$\frac{\Delta\text{ Surface Area}}{\text{Surface Area}} = \frac{13.82}{6(33.2)^2} = 0.2\%$$

so actually it's quite small.

22.19(b)

We know that the volume will change by

$$\begin{aligned}
\Delta V &= 3\alpha V \Delta T \\
\Delta V &= (3)(19 \times 10^{-6})(33.2^3)(55) \\
\Delta V &= 114.72\ cm^3
\end{aligned}$$

Check the fractional change in volume

$$\frac{\Delta V}{V} = \frac{114.72}{(33.2)^3} = 0.31\%$$

Again, quite a small change.

HRK 22.22

The length of the portion of the rod that is being heated, will change like

$$\Delta L = \alpha L \Delta T.$$

As the length increases the radioactive source moves by the same distance ΔL. We want the rate of change of displacement of the source to be 96 nm/s i.e.

$$\frac{\Delta L}{\text{second}} = 96 \times 10^{-9}\ m/s.$$

$$\begin{aligned}
\frac{\alpha L \Delta T}{\text{second}} &= 96 \times 10^{-9}\ m/s. \\
\frac{\Delta T}{\text{second}} &= \frac{96 \times 10^{-9}\ m/s}{\alpha L} \\
\frac{\Delta T}{\text{second}} &= \frac{96 \times 10^{-9}\ m/s}{(23 \times 10^{-6} K^{-1})(1.8 \times 10^{-2} m)} \\
\frac{\Delta T}{\text{second}} &= 0.2318\ K/s
\end{aligned}$$

HRK 22.30

$$\begin{aligned}
\Delta A &= A_f - A_i \\
\Delta A &= (a + \Delta a)(b + \Delta b) - ab \\
\Delta A &= ab + a\Delta b + b\Delta a + \Delta a \Delta b - ab \\
\Delta A &= a\Delta b + b\Delta a + \underbrace{\Delta a \Delta b}_{\approx 0} \\
\Delta A &= a\Delta b + b\Delta a \\
\Delta A &= a(\alpha b \Delta T) + b\Delta(\alpha a \Delta T) \quad \text{using } \Delta b = \alpha b \Delta T \text{ etc.} \\
\Delta A &= 2(ab)(\alpha \Delta T) \\
\Delta A &= 2\alpha A \Delta T
\end{aligned}$$

HRK 22.31

$$
\begin{aligned}
\Delta V &= V_f - V_i \\
\Delta V &= (a+\Delta a)(b+\Delta b)(c+\Delta c) - abc \\
\Delta V &= abc + a\Delta bc + b\Delta ac + \Delta a\Delta bc - abc \\
&+ ab\Delta c + a\Delta b\Delta c + b\Delta a\Delta c + \Delta a\Delta b\Delta c - ab\Delta c \\
\Delta V &= ab\Delta c + a\Delta bc + \Delta abc + \underbrace{a\Delta b\Delta c + \Delta ab\Delta c + a\Delta b\Delta c + \Delta a\Delta b\Delta c}_{\approx 0} \\
\Delta V &= ab\Delta c + a\Delta bc + \Delta abc \\
\Delta V &= ab(\alpha c\Delta T) + a(\alpha b\Delta T)c + (\alpha a\Delta T)bc \quad \text{using } \Delta b = \alpha b\Delta T \text{ etc.} \\
\Delta V &= 3(abc)(\alpha\Delta T) \\
\Delta V &= 3\alpha V\Delta T
\end{aligned}
$$

HRK 22.39

22.39(a)

The cylinder is rotating, and not acted on by any external torques so that the angular momentum L_z is constant.

22.39(b)

The expansion causes a redistribution of mass in the cylinder, so although L_z is constant, the moment of inertia I and the radius of the cylinder r change.

$$
\begin{aligned}
L_z = I_i\omega_i &= I_f\omega_f \qquad (\textbf{i}\text{nitial, }\textbf{f}\text{inal}) \\
\Rightarrow \frac{I_i}{I_f} &= \frac{\omega_f}{\omega_i}
\end{aligned}
$$

Recall from your study of rotation of rigid bodies that $I = \frac{1}{2}Mr^2$ so

$$
\begin{aligned}
I_i \propto r_i^2 \text{ and } I_f \propto r_f^2 \\
\Rightarrow \frac{I_i}{I_f} = \frac{r_i^2}{r_f^2}
\end{aligned}
$$

Since we are that told the radius increased by 0.18%, that means

$$
\begin{aligned}
\frac{r_i}{r_f} &= \frac{1}{1.0018} \\
\Rightarrow \frac{I_i}{I_f} &= \left(\frac{1}{1.0018}\right)^2 = \frac{\omega_f}{\omega_i}
\end{aligned}
$$

Now we know enough to calculate the percentage change in angular velocity $\frac{\Delta\omega}{\omega_i}$:

$$
\begin{aligned}
\frac{\omega_f}{\omega_i} &= \left(\frac{1}{1.0018}\right)^2 \\
\frac{\omega_f}{\omega_i} &= 0.9964 \\
\frac{\omega_f}{\omega_i} &= (1 - .0036)
\end{aligned}
$$

which implies a decrease of 0.36%.

22.39(c)

Recall that rotational kinetic energy is given by

$$K^{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}L\omega$$

where we use the second form because we know that L does not change during the expansion. To calculate the percentage change in rotational kinetic energy we use

$$\begin{aligned}
\frac{\Delta K^{rot}}{K_i^{rot}} &= \left(\frac{\frac{1}{2}L(\Delta\omega)}{\frac{1}{2}L(\omega_i)}\right) \\
\frac{\Delta K^{rot}}{K_i^{rot}} &= \left(\frac{\Delta\omega}{\omega_i}\right) \\
\frac{\Delta K^{rot}}{K_i^{rot}} &= -0.0036
\end{aligned}$$

Since we already calculated the value for $\frac{\Delta\omega}{\omega}$ in part (b).

HRK 22.41

Method 1

From chapter 15 we had

$$\begin{aligned}
P &= 2\pi\sqrt{\frac{I}{Mgd}} \\
\Rightarrow P &= kI^{\frac{1}{2}}d^{-\frac{1}{2}}
\end{aligned}$$

where in the second form we just group all the constants into one constant k.

From the previous question we know that expansion/contraction will affect the moment of Inertia I. We also know it will affect the distance from center-of-mass to pivot point d. What is the combined effect of these two processes?

$$\begin{aligned}
\delta P &= \frac{\partial P}{\partial I}\delta I + \frac{\partial P}{\partial d}\delta d \\
\delta P &= \left(\frac{k}{2}\right) I^{-\frac{1}{2}}d^{-\frac{1}{2}}\delta I + \left(-\frac{k}{2}\right) I^{-\frac{1}{2}}d^{-\frac{3}{2}}\delta d \\
\Rightarrow \frac{\delta P}{P} &= \frac{\left(\frac{k}{2}\right) I^{-\frac{1}{2}}d^{-\frac{1}{2}}\delta I + \left(-\frac{k}{2}\right) I^{-\frac{1}{2}}d^{-\frac{3}{2}}\delta d}{kI^{\frac{1}{2}}d^{-\frac{1}{2}}} \\
\frac{\delta P}{P} &= \frac{1}{2}\frac{\delta I}{I} - \frac{1}{2}\frac{\delta d}{d}
\end{aligned}$$

Now concentrate on the change in moment of inertia

$$\begin{aligned}
I &= \sum_i m_i r_i^2 \\
\delta I &= 2\sum m_i r_i \delta r_i \\
\delta I &= 2\sum m_i r_i^2 \frac{\delta r_i}{r_i} \quad (r_i \text{ are lines which expand/contract}) \\
\delta I &= 2\sum m_i r_i^2 (\alpha\Delta T) \\
\delta I &= 2I(\alpha\Delta T) \\
\frac{\delta I}{I} &= 2(\alpha\Delta T)
\end{aligned}$$

$$\begin{aligned}
\frac{\delta P}{P} &= \frac{1}{2}\frac{\delta I}{I} - \frac{1}{2}\frac{\delta d}{d} \\
\frac{\delta P}{P} &= \frac{1}{2}(2\alpha\Delta T) - \frac{1}{2}\frac{\delta d}{d} \\
\frac{\delta P}{P} &= \alpha\Delta T - \frac{1}{2}\alpha\Delta T \quad (\text{using } \frac{\delta d}{d} = \alpha\Delta T) \\
\frac{\delta P}{P} &= \frac{1}{2}\alpha\Delta T
\end{aligned}$$

All that is left is to plug in numbers

$$\begin{aligned}
\frac{\Delta P}{P} &= \frac{\alpha\Delta T}{2} \\
\frac{\Delta P}{P} &= \frac{19\times 10^{-6}(-20)}{2} \\
\frac{\Delta P}{P} &= -1.9\times 10^{-4}
\end{aligned}$$

If P is 1 hour then the clock's period changes (decreases actually) by 1.9×10^{-4} times 1 hour.

$$\Delta P = (1.9\times 10^{-4})(3600) = 0.684\ s$$

Method 2

Recall, the period P for a physical pendulum can be written as

$$P = 2\pi\sqrt{\frac{L}{g}}$$

where L is a length involving the moment of inertia (I), mass (M) and distance (d) from center-of-mass to pivot

$$L = \frac{I}{Md}.$$

$$\begin{aligned}
\Delta P &= \frac{\partial P}{\partial L}\Delta L \\
\Delta P &= \frac{\partial}{\partial L}\left(2\pi\sqrt{\frac{L}{g}}\right)\Delta L \\
\Delta P &= \left(\pi\sqrt{\frac{1}{Lg}}\right)\Delta L
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta P}{P} &= \left(\frac{\pi}{\sqrt{Lg}}\right)\Delta L\left(\frac{\sqrt{g}}{2\pi\sqrt{L}}\right) \\
\Rightarrow \frac{\Delta P}{P} &= \frac{\Delta L}{2L}
\end{aligned}$$

Finally, substitute the familiar expression for $\frac{\Delta L}{L} = \alpha\Delta T$ to get

$$\begin{aligned}
\frac{\Delta P}{P} &= \frac{\alpha\Delta T}{2} \\
\Delta P &= \frac{1}{2}\alpha P\Delta T
\end{aligned}$$

which is the expression you are asked to derive in question 40.

All that is left is to plug in numbers

$$\begin{aligned}
\frac{\Delta P}{P} &= \frac{\alpha \Delta T}{2} \\
\frac{\Delta P}{P} &= \frac{19 \times 10^{-6}(-20)}{2} \\
\frac{\Delta P}{P} &= -1.9 \times 10^{-4}
\end{aligned}$$

If P is 1 hour then the clock's period changes (decreases actually) by 1.9×10^{-4} times 1 hour.

$$\Delta P = (1.9 \times 10^{-4})(3600) = 0.684\ s$$

Multiple Integral Problems

Q. 1

As suggested in the question, we need to find the equation of a line (i.e. ($x(y)$: x as a function of y) for each of the two lines that intersect to form the apex of the triangle. The line on the left goes from ($x = -a, y = 0$) to ($x = 0, y = c$). The line on the right goes from ($x = b, y = 0$) to ($x = 0, y = c$).

$$\begin{aligned}
\text{left: } x(y) &= \frac{a}{c}y - a \\
\text{right: } x(y) &= -\frac{b}{c}y + b
\end{aligned}$$

This tells us our limits of integration

$$\begin{aligned}
A &= \int\int dxdy \\
A &= \int \left[\int_{left}^{right} dx\right] dy \\
A &= \int \left[\int_{\frac{a}{c}y-a}^{-\frac{b}{c}y+b} dx\right] dy \\
A &= \int \left[x\Big]_{\frac{a}{c}y-a}^{-\frac{b}{c}y+b}\ \right] dy \\
A &= \int \left[-\frac{b}{c}y + b - \frac{a}{c}y + a\right] dy \\
A &= \int \left[-\frac{b}{c}y + b - \frac{a}{c}y + a\right] dy \\
A &= \int \left[(a+b) - \frac{(a+b)}{c}y\right] dy
\end{aligned}$$

The quantity in the square brackets is the length of a horizontal line inside the triangle, at a height y above the base. Integrate this quantity over y values from the bottom to the top of the triangle to get the area.

$$\begin{aligned}
A &= \int_0^c \left((a+b) - \frac{(a+b)}{c}y\right) dy \\
A &= (a+b)y - \frac{(a+b)}{2c}y^2\Big]_0^c \\
A &= (a+b)c - \frac{(a+b)}{2c}c^2 \\
A &= \frac{(a+b)c}{2}
\end{aligned}$$

which is your normal, half-base-times-height expression.

Q. 2

We will do this two ways. If the density was uniform, we could find the mass M by multiplying ρ and V. Since the density varies with x and y, we must integrate $\rho(x, y)$ over the area of the plate. Since the expression for the density does not depend on t, we do not need to integrate over z – we merely need to multiply the thickness t by the mass for each infinitesimal slice in the $x - y$ plane.

We don't *need to* integrate over the vertical coordinate z, but we can, as ultimately it amounts to multiplying by t anyway.

Method 1 (Double Integral):

$$
\begin{aligned}
M &= \left(\int\int \rho(x,y)\ dxdy\right) \times t \\
M &= \left(\int_{y=0}^{y=b}\int_{x=0}^{x=a} (\rho_0 + \rho_1 xy)\ dxdy\right) \times t \\
M &= \left(\int_{y=0}^{y=b}\left[\int_{x=0}^{x=a} (\rho_0 + \rho_1 xy)\ dx\right] dy\right) \times t \\
M &= \left(\int_{y=0}^{y=b}\left[\left(\rho_0 x + \rho_1 \frac{x^2 y}{2}\right)\Big]_0^a\right] dy\right) \times t \\
M &= \left(\int_{y=0}^{y=b}\left[\left(\rho_0 a + \rho_1 \frac{a^2 y}{2}\right)\right] dy\right) \times t
\end{aligned}
$$

Now that we have integrated over x move on to y

$$
\begin{aligned}
M &= \left(\left[\int_{y=0}^{y=b}\left(\rho_0 a + \rho_1 \frac{a^2 y}{2}\right) dy\right]\right) \times t \\
M &= \left(\left[\left(\rho_0 a y + \rho_1 \frac{a^2 y^2}{4}\right)\Big]_0^b\right]\right) \times t \\
M &= \left(\left[\left(\rho_0 ab + \rho_1 \frac{a^2 b^2}{4}\right)\right]\right) \times t \\
M &= \left(\rho_0 abt + \rho_1 \frac{a^2 b^2 t}{4}\right)
\end{aligned}
$$

Method 2 (Triple Integral):

Integrate the density over the volume to get mass

$$
\begin{aligned}
M &= \int\int\int \rho(x,y)\ dxdydz \\
M &= \int_{z=0}^{z=t}\int_{y=0}^{y=b}\int_{x=0}^{x=a} (\rho_0 + \rho_1 xy)\ dxdydz \\
M &= \int_{z=0}^{z=t}\int_{y=0}^{y=b}\left[\int_{x=0}^{x=a} (\rho_0 + \rho_1 xy)\ dx\right] dydz \\
M &= \int_{z=0}^{z=t}\int_{y=0}^{y=b}\left[\left(\rho_0 x + \rho_1 \frac{x^2 y}{2}\right)\Big]_0^a\right] dydz \\
M &= \int_{z=0}^{z=t}\int_{y=0}^{y=b}\left[\left(\rho_0 a + \rho_1 \frac{a^2 y}{2}\right)\right] dydz
\end{aligned}
$$

Now that we have integrated over x move on to y

$$\begin{aligned}
M &= \int_{z=0}^{z=t} \left[\int_{y=0}^{y=b} \left(\rho_0 a + \rho_1 \frac{a^2 y}{2} \right) dy \right] dz \\
M &= \int_{z=0}^{z=t} \left[\left(\rho_0 a y + \rho_1 \frac{a^2 y^2}{4} \right) \Big]_0^b \right] dz \\
M &= \int_{z=0}^{z=t} \left[\left(\rho_0 a b + \rho_1 \frac{a^2 b^2}{4} \right) \right] dz
\end{aligned}$$

Finally, integrate over z

$$\begin{aligned}
M &= \int_{z=0}^{z=t} \left(\rho_0 a b + \rho_1 \frac{a^2 b^2}{4} \right) dz \\
M &= \left(\rho_0 a b z + \rho_1 \frac{a^2 b^2 z}{4} \right) \Big]_0^t \\
M &= \left(\rho_0 a b t + \rho_1 \frac{a^2 b^2 t}{4} \right)
\end{aligned}$$

Units of ρ_1

We can examine the units of this quantity to see what units ρ_1 has

$$\begin{aligned}
[M] &= [\rho_0][abt] + [\rho_1][\frac{a^2 b^2 t}{4}] \\
[kg] &= [\frac{kg}{m^3}][m^3] + [\rho_1][m^5] \\
\Rightarrow [\rho_1] &= \frac{kg}{m^5}
\end{aligned}$$

These units are equivalent to density over area.

Q. 3

Q. 3 (Cartesian)

In Cartesian co-ordinates the equation of a circle is $x^2 + y^2 = R^2$. Similar to the triangle in the first question, we can get the x coordinate of the boundary for a given value of y. It's a little bit simpler to just concentrate on one quarter of the circle (say, the positive quadrant $x \geq 0, y \geq 0$) and realize that the area of the circle is four time the area of this region. With this region in mind, the left boundary is given by $x = 0$ and the right boundary is given by $x(y) = \sqrt{R^2 - y^2}$.

$$\begin{aligned}
A &= \int\int dx dy \\
A &= \int \left[\int_{left}^{right} dx \right] dy \\
A &= \int \left[\int_{x=0}^{x=\sqrt{R^2-y^2}} dx \right] dy \\
A &= \int \left[x \Big]_{x=0}^{x=\sqrt{R^2-y^2}} \right] dy \\
A &= \int \left[\sqrt{R^2 - y^2} \right] dy
\end{aligned}$$

Now integrate form the bottom of the quadrant $y = 0$ to the top $y = R$

$$
\begin{aligned}
A &= \int_0^R \sqrt{R^2 - y^2}\, dy \qquad \text{(use a trig substitution } y = R\sin\theta) \\
A &= \int_0^R \sqrt{R^2(1-\sin^2\theta)}\, dy \qquad \text{(and noting that } dy = R\cos\theta d\theta) \\
A &= \int_{\theta=0}^{\theta=\pi/2} (R\cos\theta) R\cos\theta d\theta \qquad \text{(changing the limits too)} \\
A &= R^2 \int_{\theta=0}^{\theta=\pi/2} \cos^2\theta d\theta \\
A &= \frac{\pi R^2}{4} \qquad \text{(recall that integrating } \cos^2\theta \text{ over a full cycle gives a value of } 2\pi)
\end{aligned}
$$

Finally, the area of a circle is 4 times the area of a quadrant so

$$A_{circle} = \pi R^2$$

Q. 3 (Polar)

Once we write the correct expression for an area element in polar coordinates, it is simply a matter of integrating over an angle of 2π and from $r = 0$ to $r = R$.

$$
\begin{aligned}
A &= \int dA \\
A &= \int\int r dr d\phi \qquad \text{(Area element in polar coordinates)} \\
A &= \int_0^{2\pi} \left[\int_0^R r dr \right] d\phi \\
A &= \int_0^{2\pi} \left[\frac{r^2}{2} \Big]_0^R \right] d\phi \\
A &= \int_0^{2\pi} \frac{R^2}{2} d\phi \\
A &= 2\pi \frac{R^2}{2} \\
A &= \pi R^2
\end{aligned}
$$

Q. 4

Q. 4 (a)

If density was constant then we would have $M = \rho V$. Since ρ is a function of r and ϕ we must integrate over those coordinates, and finally multiply by the thickness t. As we discussed in Q.2, we could just as easily integrate over the z coordinate too, instead of multiplying by the thickness – either way it amounts to the same thing. The nice thing about doing it with a triple integral is that we can see how we would do the problem if ρ happened to be a function of z too i.e. if we had $\rho(r, \phi, z)$.

$$
\begin{aligned}
M &= \int_V \rho dV \\
M &= \int_{z=0}^{z=t} \int_0^{2\pi} \int_0^R \rho(r,\phi)\ r dr d\phi dz \\
M &= \int_{z=0}^{z=t} \int_0^{2\pi} \left[\int_0^R \left(\rho_0 + \rho_1 r^2 \sin(\phi/2)\right) r dr \right] d\phi dz \\
M &= \int_{z=0}^{z=t} \int_0^{2\pi} \left[\int_0^R \left(\rho_0 r + \rho_1 r^3 \sin(\phi/2)\right) dr \right] d\phi dz \\
M &= \int_{z=0}^{z=t} \int_0^{2\pi} \left[\left(\rho_0 \frac{r^2}{2} + \rho_1 \frac{r^4}{4} \sin(\phi/2)\right) \Big]_0^R \right] d\phi dz \\
M &= \int_{z=0}^{z=t} \int_0^{2\pi} \left[\left(\rho_0 \frac{R^2}{2} + \rho_1 \frac{R^4}{4} \sin(\phi/2)\right) \right] d\phi dz \\
M &= \int_{z=0}^{z=t} \left[\int_0^{2\pi} \left(\rho_0 \frac{R^2}{2} + \rho_1 \frac{R^4}{4} \sin(\phi/2)\right) d\phi \right] dz \\
M &= \int_{z=0}^{z=t} \left[\left(\rho_0 \frac{R^2}{2}\phi + \rho_1 \frac{R^4}{4}(-2\cos(\phi/2))\right) \Big]_0^{2\pi} \right] dz \\
M &= \int_{z=0}^{z=t} \left[\left(\rho_0 \frac{R^2}{2} 2\pi + \rho_1 \frac{R^4}{4}(-2\cos(\pi))\right) - \left(0 + \rho_1 \frac{R^4}{4}(-2\cos(0))\right) \right] dz \\
M &= \int_{z=0}^{z=t} \left[\left(\rho_0 \pi R^2 + \rho_1 \frac{R^4}{2}\right) - \left(0 - \rho_1 \frac{R^4}{2}\right) \right] dz \\
M &= \int_{z=0}^{z=t} \left[\rho_0 \pi R^2 + \rho_1 R^4\right] dz \quad \text{(remember this last integration is equivalent to multiplying by } t\text{)} \\
M &= \rho_0 \pi R^2 z + \rho_1 R^4 z \Big]_0^t \\
M &= \rho_0 \pi R^2 t + \rho_1 R^4 t
\end{aligned}
$$

You can see that the units work out if ρ_1 has units of density/area as before.

Q. 4 (b)

$$I = \int dI = \int r^2\ dm = \int_V r^2 \rho\ dV$$

$$
\begin{aligned}
I &= \int_{z=0}^{z=t}\int_0^{2\pi}\int_0^R r^2\rho(r,\phi)\; rdrd\phi dz \\
I &= \int_{z=0}^{z=t}\int_0^{2\pi}\left[\int_0^R r^2\left(\rho_0+\rho_1 r^2\sin(\phi/2)\right) rdr\right] d\phi dz \\
I &= \int_{z=0}^{z=t}\int_0^{2\pi}\left[\int_0^R \left(\rho_0 r^3+\rho_1 r^5\sin(\phi/2)\right) dr\right] d\phi dz \\
I &= \int_{z=0}^{z=t}\int_0^{2\pi}\left[\left(\rho_0\frac{r^4}{4}+\rho_1\frac{r^6}{6}\sin(\phi/2)\right)\Big]_0^R\right] d\phi dz \\
I &= \int_{z=0}^{z=t}\int_0^{2\pi}\left[\left(\rho_0\frac{R^4}{4}+\rho_1\frac{R^6}{6}\sin(\phi/2)\right)\right] d\phi dz \\
I &= \int_{z=0}^{z=t}\left[\int_0^{2\pi}\left(\rho_0\frac{R^4}{4}+\rho_1\frac{R^6}{6}\sin(\phi/2)\right) d\phi\right] dz \\
I &= \int_{z=0}^{z=t}\left[\left(\rho_0\frac{R^4}{4}\phi+\rho_1\frac{R^6}{6}(-2\cos(\phi/2))\right)\Big]_0^{2\pi}\right] dz \\
I &= \int_{z=0}^{z=t}\left[\left(\rho_0\frac{R^4}{4}2\pi+\rho_1\frac{R^6}{6}(-2\cos(\pi))\right)-\left(0+\rho_1\frac{R^6}{6}(-2\cos(0))\right)\right] dz \\
I &= \int_{z=0}^{z=t}\left[\left(\rho_0\pi\frac{R^4}{2}+\rho_1\frac{R^6}{3}\right)-\left(0-\rho_1\frac{R^6}{3}\right)\right] dz \\
I &= \int_{z=0}^{z=t}\left[\rho_0\pi\frac{R^4}{2}+\rho_1\frac{2R^6}{3}\right] dz \\
I &= \frac{1}{2}\rho_0\pi R^4 z+\frac{2}{3}\rho_1 R^6 z\Big]_0^t \\
I &= \frac{1}{2}\rho_0\pi R^4 t+\frac{2}{3}\rho_1 R^6 t
\end{aligned}
$$

Q. 4 (c)

Divide I of part (b) by M of part (a) to get R_G^2

$$
R_G^2 = \frac{\frac{1}{2}\rho_0\pi R^4 t+\frac{2}{3}\rho_1 R^6 t}{\rho_0\pi R^2 t+\rho_1 R^4 t}
$$

$$
R_G^2 = R^2\left(\frac{\frac{1}{2}\rho_0\pi R^2 t+\frac{2}{3}\rho_1 R^4 t}{\rho_0\pi R^2 t+\rho_1 R^4 t}\right)
$$