HRK 24.5

First we need to find the number of molecules per unit volume N/V, which is also called ρ_n (number per volume instead of mass per volume like normal ρ).

24.5 (a)

$$\begin{aligned}
\frac{N}{V} &= \frac{P}{k_B T} \\
\frac{N}{V} &= \frac{1.01 \times 10^5}{(1.38 \times 10^{-23}(273)} \\
\rho_n = \frac{N}{V} &= 2.68 \times 10^{25} \text{ molecules}/m^3
\end{aligned}$$

24.5 (b)

Since

$$\begin{aligned}
\lambda &= \frac{1}{\sqrt{2}\,\pi \rho_n d^2} \\
\lambda &= \frac{1}{\sqrt{2}\,\pi \left(\frac{N}{V}\right) d^2}
\end{aligned}$$

then we have

$$\begin{aligned}
d &= \frac{1}{(\sqrt{2}\,\pi \left(\frac{N}{V}\right) \lambda)^{\frac{1}{2}}} \\
d &= \frac{1}{(\sqrt{2}\,\pi (2.68 \times 10^{25})(285 \times 10^{-9}))^{\frac{1}{2}}} \\
d &= 1.72 \times 10^{-10}\ m \\
d &= 1.72\ A^\circ \text{ (Angstrom} = 10^{-10}\ m)
\end{aligned}$$

The Angstrom is a common and usefel unit when dealing with the atomic realm!

HRK 24.12

24.12 (a)

$$\begin{aligned}
\langle v \rangle = \overline{v} &= \frac{2+3+4+5+6+7+8+9+10+11}{10} \\
&= \frac{65}{10} = 6.5\ m/s
\end{aligned}$$

24.12 (b)

$$\begin{aligned}
\langle v^2 \rangle = \overline{v^2} &= \frac{2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2}{10} \\
&= \frac{505}{10} = 50.5\ m^2/s^2
\end{aligned}$$

$$\begin{aligned}
v_{rms} &= \sqrt{\langle v^2 \rangle} \\
v_{rms} &= 7.11\ m/s
\end{aligned}$$

HRK 24.15

The way this velocity selector works is that the cylinder must rotate through an angle θ in the time it takes a molecule of the desired speed to travel the length, L, of the cylinder. That time is $\delta t = \frac{L}{r}$ so we have

$$\begin{aligned}
\frac{L}{v} &= \frac{\phi}{\omega} \\
\Rightarrow \frac{0.204}{212} &= \frac{0.0841}{\omega} \\
\Rightarrow \omega &= \frac{0.0841(212)}{0.204} \\
\omega &= 87.4\ rad\ s^{-1} \\
\omega &= 13.9(2\pi)\ rad\ s^{-1} \\
\omega &= 13.9\ rev\ s^{-1} \\
\omega &= 834\ rpm
\end{aligned}$$

843 revolutions per minute is easily achieved – most electric motors run at 1800 rpm.

HRK 24.21

24.21 (a)

It must be true that if we add up (or in this case integrate), the number of particles with a certain speed, over all the different possible speeds, we should get the total number of particles N

$$\begin{aligned}
N &= \int_{-\infty}^{\infty} n(v)\ dv \quad \text{(but } n(v) = 0 \text{ outside range } 0 \leq v \leq v_0 \text{ so..)} \\
N &= \int_0^{v_0} n(v)\ dv \\
N &= \int_0^{v_0} Cv^2\ dv \\
N &= C\frac{v^3}{3}\Big]_0^{v_0} \\
N &= C\frac{v_0^3}{3} \\
\Rightarrow C &= \frac{3N}{v_0^3} \left(\text{units: } \frac{s^3}{m^3}\right)
\end{aligned}$$

24.21 (b)

$$\begin{aligned}
\langle v \rangle &= \frac{1}{N}\int_0^{v_0} (v)Cv^2\ dv \\
\langle v \rangle &= \frac{C}{N}\int_0^{v_0} v^3\ dv \\
\langle v \rangle &= \frac{C}{N}\frac{v^4}{4}\Big]_0^{v_0} \\
\langle v \rangle &= \frac{C}{N}\frac{v_0^4}{4} \\
\langle v \rangle &= \frac{3N}{Nv_0^3}\frac{v_0^4}{4} \\
\langle v \rangle &= \frac{3}{4}v_0\ m/s
\end{aligned}$$

24.21 (c)

$$\begin{aligned}
\langle v^2 \rangle &= \frac{1}{N}\int_0^{v_0} (v^2)Cv^2\ dv \\
\langle v^2 \rangle &= \frac{1}{N}\int_0^{v_0} Cv^4\ dv \\
\langle v^2 \rangle &= \frac{C}{N}\frac{v^5}{5}\Big]_0^{v_0} \\
\langle v^2 \rangle &= \frac{C}{N}\frac{v_0^5}{5} \\
\langle v^2 \rangle &= \frac{3N}{Nv_0^3}\frac{v_0^5}{5} \\
\langle v^2 \rangle &= \frac{3}{5}v_0^2\ m^2/s^2 \\
\Rightarrow v_{rms} &= \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3}{5}}\ v_0\ m/s
\end{aligned}$$

HRK 24.22

It is easy to construct the function $n(v)$ just by looking at the graph of $n(v)$ vs. v.

$$\begin{aligned}
n(v) &= \frac{a}{v_0}v \quad 0 \le v \le v_0 \\
n(v) &= a \qquad v_0 \le v \le 2v_0 \\
n(v) &= 0 \ \ \text{otherwise}
\end{aligned}$$

24.22 (a)

The total number of particles N, is found by getting the area under the $n(v)$ curve.

General way:

$$\begin{aligned}
N &= \int_{-\infty}^{\infty} n(v)\ dv \\
N &= \int_0^{v_0} \frac{a}{v_0}v\ dv + \int_{v_0}^{2v_0} a\ dv \\
N &= \left(\frac{a}{v_0}\right)\frac{v^2}{2}\Big]_0^{v_0} + (a)v\Big]_{v_0}^{2v_0} \\
N &= \frac{av_0}{2} + av_0 \\
N &= \frac{3av_0}{2} \\
\Rightarrow a &= \frac{2N}{3v_0}
\end{aligned}$$

Shortcut for this question:

Since the function is so simple, we don't *need* to integrate; we can just use areas of triangles and rectangles.

$$\begin{aligned} N &= \frac{1}{2}av_0 + av_0 \\ N &= \frac{3}{2}av_0 \\ \Rightarrow a &= \frac{2N}{3v_0} \end{aligned}$$

24.22 (b)

General way:

$$\begin{aligned} N(1.5v_0 \le v \le 2v_0) &= \int_{1.5v_0}^{2v_0} a\ dv \\ N(1.5v_0 \le v \le 2v_0) &= (a)v\Big]_{1.5v_0}^{2v_0} \\ N(1.5v_0 \le v \le 2v_0) &= \frac{av_0}{2} \\ N(1.5v_0 \le v \le 2v_0) &= \frac{v_0}{2}\frac{2N}{3v_0} \\ N(1.5v_0 \le v \le 2v_0) &= \frac{N}{3} \end{aligned}$$

Shortcut for this question:

Instead of integrating from 1.50 v_0 to 2.00 v_0, we will just get the area of the corresponding rectangle.

$$\begin{aligned} Area &= \frac{v_0}{2} \times a \\ Area &= \frac{v_0}{2}\frac{2N}{3v_0} \\ Area &= \frac{N}{3} \end{aligned}$$

This is the ratio of the rectangle of interest, compared to the whole area under the curve, times N.

24.22 (c)

Again, $n(v)$ is zero outside the range $0 \le x \le 2v_0$ so we need only integrate over that region. We will integrate v times $n(v)$, being careful to use the correct form for $n(v)$ for the relevant range of integration. We need to divide the integral by N because $n(v)dv$ is the *number* of molecules having speed between v and $v + dv$.

$$\begin{aligned}
\langle v \rangle &= \frac{1}{N}\left(\int_0^{v_0}(v)n(v)\ dv + \int_{v_0}^{2v_0}(v)n(v)\ dv\right)\\
\langle v \rangle &= \frac{1}{N}\left(\int_0^{v_0}(v)\frac{a}{v_0}v\ dv + \int_{v_0}^{2v_0}(v)a\ dv\right)\\
\langle v \rangle &= \frac{1}{N}\left(\int_0^{v_0}\frac{a}{v_0}v^2\ dv + \int_{v_0}^{2v_0}av\ dv\right)\\
\langle v \rangle &= \frac{1}{N}\left(\frac{a}{v_0}\frac{v^3}{3}\Big]_0^{v_0} + a\frac{v^2}{2}\Big]_{v_0}^{2v_0}\right)\\
\langle v \rangle &= \frac{1}{N}\left(\frac{a}{v_0}\frac{v_0^3}{3} + a\frac{3v_0^2}{2}\right)\\
\langle v \rangle &= \frac{1}{N}\left(\frac{av_0^2}{3} + \frac{3av_0^2}{2}\right)\\
\langle v \rangle &= \frac{1}{N}\left(\frac{a11v_0^2}{6}\right)\\
\langle v \rangle &= \frac{1}{N}\left(\frac{2N}{3v_0}\right)\frac{11v_0^2}{6}\\
\langle v \rangle &= \frac{11v_0}{9}\ m/s
\end{aligned}$$

24.22 (d)

We will integrate v^2 times $n(v)$, being careful to use the correct form for $n(v)$ for the relevant range of integration.

$$\begin{aligned}
\langle v^2 \rangle &= \frac{1}{N}\left(\int_0^{v_0}(v^2)n(v)\ dv + \int_{v_0}^{2v_0}(v^2)n(v)\ dv\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\int_0^{v_0}(v^2)\frac{a}{v_0}v\ dv + \int_{v_0}^{2v_0}(v^2)a\ dv\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\int_0^{v_0}\frac{a}{v_0}v^3\ dv + \int_{v_0}^{2v_0}av^2\ dv\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\frac{a}{v_0}\frac{v^4}{4}\Big]_0^{v_0} + a\frac{v^3}{3}\Big]_{v_0}^{2v_0}\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\frac{a}{v_0}\frac{v_0^4}{4} + a\frac{7v_0^2}{3}\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\frac{av_0^3}{4} + \frac{7av_0^2}{3}\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\frac{a31v_0^3}{12}\right)\\
\langle v^2 \rangle &= \frac{1}{N}\left(\frac{2N}{3v_0}\right)\frac{31v_0^3}{12}\\
\langle v^2 \rangle &= \frac{31v_0^2}{18}\ m^2/s^2\\
\Rightarrow v_{rms} = \sqrt{\langle v^2 \rangle} &= \sqrt{\frac{31}{18}}\ v_0\ m/s
\end{aligned}$$

StatMech 1

1 (a)

We know that probabilities must sum to one, when all possible events are taken into account. Using a continuous probability distribution, this means that the integral of the distribution over all possibilities must evaluate to one.

$$
\begin{aligned}
\int_{-\infty}^{\infty} w(x)\ dx &= 1 \\
\Rightarrow C\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} &= 1 \\
C\int_{-\infty}^{\infty} e^{-a^2x^2} &= 1 \qquad \text{where } a = \frac{1}{\sqrt{2}\ \sigma} \\
C\frac{\sqrt{\pi}}{a} &= 1 \\
C\sqrt{2\pi}\sigma &= 1 \\
\Rightarrow C &= \frac{1}{\sqrt{2\pi}\sigma}
\end{aligned}
$$

1 (b)

Recall from your notes, the technique for finding the average value of some function $f(x)$ is to integrate over the probability distribution $w(x)$ like so

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)w(x)\ dx.$$

Here we want to find the expected value for position $\langle x\rangle$ and so the relevant integral is

$$\langle x\rangle \quad = \quad \int_{-\infty}^{\infty} (x)e^{-\frac{x^2}{2\sigma^2}}\ dx.$$

Although the question gives you two standard integrals that you can use,

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-a^2x^2} &= \frac{\sqrt{\pi}}{a} \\
\int_{-\infty}^{\infty} x^2e^{-a^2x^2} &= \frac{\sqrt{\pi}}{2a^3}
\end{aligned}
$$

neither of these is exactly of the form we want.

In your notes on distributions, it was noted that

$$p(x) \sim e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

corresponds to a Gaussian distribution centered at $x = x_0$. The particular distribution in this question corresponds to this when we set $x_0 = 0$. Physically, it should seem reasonable that if the most probable value of x is zero, and if the distribution is symmetric about $x = 0$, then overall we will have $\langle x\rangle = 0$. To prove this mathematically we will show why the above integral for $\langle x\rangle$ evaluates to zero.

We already said that $w(x)$ is of the form $e^{-a^2x^2}$ and so it satisfies, for all values of x,

$$w(x) = w(-x) \qquad \Rightarrow w(x) = e^{-a^2x^2} \text{ is an even function}$$

If you were to draw any even function of x in the $x - y$ plane, it would be reflected (symmetric) about the y axis.

On the other hand the function $f(x) = x$ is an odd function

$$f(x) = -f(-x) \qquad \Rightarrow f(x) = x \text{ is an odd function}$$

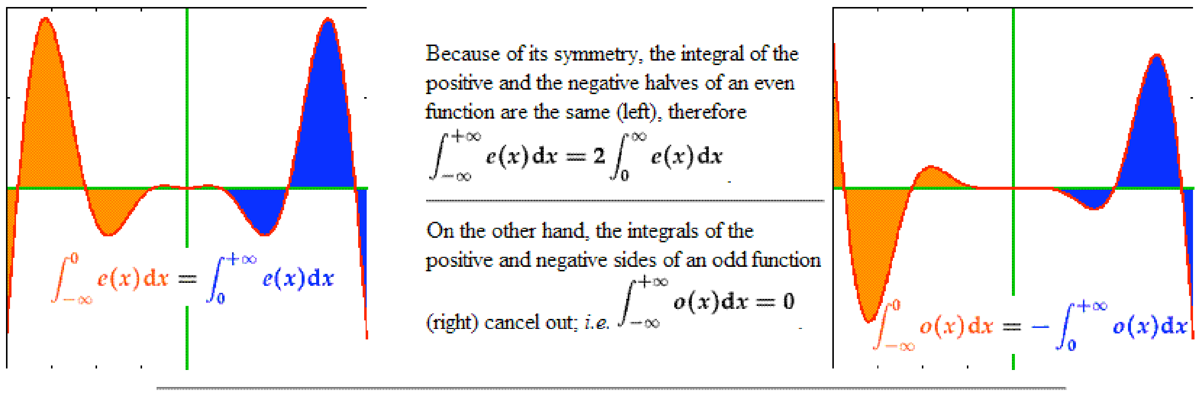

Figure 1: Integrating odd and even functions: Explanation from: `http://users.aber.ac.uk/ruw/teach/340/ft_symmetry.html`

If you were to draw any odd function of x in the $x-y$ plane, it would be flipped about both the x and the y axis (in short, it would be reflected through the origin).

The products of odd and even functions produce an overall function which is either even or odd; the rules obey the same pattern as multiplying ± 1

$$\begin{aligned}
(\text{even})(\text{even}) &= (\text{even}) \\
(\text{even})(\text{odd}) &= (\text{odd}) \\
(\text{odd})(\text{even}) &= (\text{odd}) \\
(\text{odd})(\text{odd}) &= (\text{even})
\end{aligned}$$

How does all of this help? It is fairly easy to convince yourself that integrating an odd function, over a region centered about the y-axis, always gives you zero i.e

$$\int_{-L}^{L} (\text{odd function of } x)\ dx = 0$$

In particular, for our case,

$$\begin{aligned}
\langle x \rangle &= C\int_{-\infty}^{\infty} (x)e^{-\frac{x^2}{2\sigma^2}}\ dx \\
\langle x \rangle &= C\int_{-\infty}^{\infty} (\text{odd})(\text{even})\ dx \\
\langle x \rangle &= C\int_{-\infty}^{\infty} \text{odd}\ dx \\
\langle x \rangle &= 0
\end{aligned}$$

1 (c)

$$\begin{aligned}
\langle x^2 \rangle &= C\int_{-\infty}^{\infty} (x^2)e^{-\frac{x^2}{2\sigma^2}}\ dx \\
\langle x^2 \rangle &= C\int_{-\infty}^{\infty} (x^2)e^{-a^2x^2}\ dx \quad \text{where } a = \frac{1}{\sqrt{2}\,\sigma}
\end{aligned}$$

Use the standard result

$$\begin{aligned} C\int_{-\infty}^{\infty} x^2 e^{-a^2x^2} &= C\frac{\sqrt{\pi}}{2a^3} \\ \Rightarrow \langle x^2 \rangle &= C\frac{\sqrt{\pi}}{2}(\sqrt{2}\ \sigma)^3 \\ \langle x^2 \rangle &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)\frac{\sqrt{\pi}}{2}(\sqrt{2}\ \sigma)^3 \\ \langle x^2 \rangle &= \sigma^2 \end{aligned}$$

1 (d)

The quantity $\langle (x - \langle x \rangle)^2 \rangle$ is a measure of how much the molecules spread around the average. If this quantity is small then measurements of x will be narrowly clusterd around $\langle x \rangle$.

$$\begin{aligned} \text{Root mean square spread:} \quad & \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \\ \Rightarrow \text{Root mean square spread:} \quad & \sqrt{\langle (x^2 - 2x\langle x \rangle + \langle x \rangle^2) \rangle} \\ \Rightarrow \text{Root mean square spread:} \quad & \sqrt{\langle x^2 \rangle} \\ \Rightarrow \text{Root mean square spread:} \quad & = \sigma \end{aligned}$$

StatMech 2

In this question n is a label that tells us the *energy* of a state. (Don't confuse this with number of particles). The lowest energy that is possible is

$$E_0 = \frac{1}{2}\hbar\omega \qquad \text{(when } n = 0)$$

The probability of finding the system in a particular state n is given by

$$\begin{aligned} P(n) &= Ce^{-\frac{E_n}{k_BT}} \\ \Rightarrow P(n) &= Ce^{-\frac{(n+\frac{1}{2})\hbar\omega}{k_BT}} \end{aligned}$$

I'm going to bunch a few constants together into one I call α

$$\begin{aligned} \text{let } \alpha &= \frac{\hbar\omega}{k_BT} \\ \Rightarrow P(n) &= Ce^{-(n+\frac{1}{2})\alpha} \end{aligned}$$

2 (a)

To find a value for C we use the fact that probabilities must sum to one.

$$\begin{aligned} \sum_{n=0}^{n=\infty} P(n) &= 1 \\ \sum_{n=0}^{n=\infty} Ce^{-(n+\frac{1}{2})\alpha} &= 1 \\ \left(Ce^{-(0+\frac{1}{2})\alpha} + Ce^{-(1+\frac{1}{2})\alpha} + Ce^{-(2+\frac{1}{2})\alpha} + \ldots\right) &= 1 \\ Ce^{-\frac{\alpha}{2}}\left(e^{-0(\alpha)} + e^{-1(\alpha)} + e^{-2(\alpha)} \ldots\right) &= 1 \\ Ce^{-\frac{\alpha}{2}}\left(1 + x + x^2 \ldots\right) &= 1 \qquad \text{where } x = e^{-\alpha} = e^{-\frac{\hbar\omega}{k_BT}} \end{aligned}$$

Since

$$e^{-\frac{\hbar\omega}{k_B T}} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}}} < 1$$

we can use the result that

$$\text{for } |x| < 1 \qquad (1 + x + x^2 + x^3 \ldots) = \frac{1}{1-x}$$

To see why the above expression is true consider

$$\begin{aligned} s &= 1 + x + x^2 + x^3 \ldots \\ \Rightarrow xs &= x + x^2 + x^3 + x^4 \ldots \\ \Rightarrow s - xs &= 1 \quad \text{if } x^N \to 0 \text{ as } N \to \infty \\ \text{Rearranging:} \quad s &= \frac{1}{1-x} \end{aligned}$$

So, returning to the physics,

$$\begin{aligned} Ce^{-\frac{\alpha}{2}}\left(1 + x + x^2 \ldots\right) &= 1 \qquad \text{where } x = e^{-\alpha} \\ \Rightarrow Ce^{-\frac{\alpha}{2}}\left(\frac{1}{1-x}\right) &= 1 \\ Ce^{-\frac{\alpha}{2}}\left(\frac{1}{1-e^{-\alpha}}\right) &= 1 \\ \text{tidying..} \qquad \frac{C}{e^{\frac{\alpha}{2}}} &= 1 - e^{-\alpha} \\ C &= e^{\frac{\alpha}{2}}(1 - e^{-\alpha}) \end{aligned}$$

Finally, plug this expression for C back in to our probability distribution

$$\begin{aligned} P(n) &= Ce^{-(n+\frac{1}{2})\alpha} \\ P(n) &= e^{\frac{\alpha}{2}}(1 - e^{-\alpha})e^{-(n+\frac{1}{2})\alpha} \\ P(n) &= e^{\frac{\alpha}{2}}(1 - e^{-\alpha})e^{-n\alpha}e^{-\frac{\alpha}{2}} \\ P(n) &= (1 - e^{-\alpha})e^{-n\alpha} \\ P(n) &= (1 - e^{-\frac{\hbar\omega}{k_B T}})e^{-n\frac{\hbar\omega}{k_B T}} \end{aligned}$$

Note that the "zero-point" energy $\frac{1}{2}\hbar\omega$, does not show up here.

2 (b)

$$\begin{aligned} \langle E \rangle &= \sum_{n=0}^{\infty} E_n P(n) \\ \langle E \rangle &= \sum_{n=0}^{\infty} \left[(n + \frac{1}{2})\hbar\omega\right](1 - e^{-\frac{\hbar\omega}{k_B T}})e^{-n\frac{\hbar\omega}{k_B T}} \\ \langle E \rangle &= \sum_{n=0}^{\infty} \left[(n + \frac{1}{2})\hbar\omega\right](1 - e^{-\alpha})e^{-n\alpha} \\ \langle E \rangle &= \sum_{n=0}^{\infty} \left((\hbar\omega)(1 - e^{-\alpha})ne^{-n\alpha} + (\frac{1}{2}\hbar\omega)(1 - e^{-\alpha})e^{-n\alpha}\right) \\ \langle E \rangle &= \sum_{n=0}^{\infty}(\hbar\omega)(1 - e^{-\alpha})\left(ne^{-n\alpha} + \frac{1}{2}e^{-n\alpha}\right) \\ \langle E \rangle &= (\hbar\omega)(1 - e^{-\alpha})\sum_{n=0}^{\infty}\left(ne^{-n\alpha} + \frac{1}{2}e^{-n\alpha}\right) \end{aligned}$$

Now make use of

$$\sum_{n=0}^{\infty} ne^{-n\alpha} = \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}$$
$$\text{and} \sum_{n=0}^{\infty} e^{-n\alpha} = \frac{1}{1-e^{-\alpha}}$$

$$\langle E \rangle = (\hbar\omega)(1-e^{-\alpha})\sum_{n=0}^{\infty}\left(ne^{-n\alpha}+\frac{1}{2}e^{-n\alpha}\right)$$
$$\langle E \rangle = (\hbar\omega)(1-e^{-\alpha})\left(\frac{e^{-\alpha}}{(1-e^{-\alpha})^2}+\frac{\frac{1}{2}}{1-e^{-\alpha}}\right)$$
$$\langle E \rangle = (\hbar\omega)\left(\frac{e^{-\alpha}}{(1-e^{-\alpha})}+\frac{1}{2}\right)$$
$$\langle E \rangle = (\hbar\omega)\left(\frac{e^{-\frac{\hbar\omega}{k_BT}}}{\left(1-e^{-\frac{\hbar\omega}{k_BT}}\right)}+\frac{1}{2}\right)$$
$$\langle E \rangle = \frac{\hbar\omega}{2}+\frac{\hbar\omega e^{-\frac{\hbar\omega}{k_BT}}}{\left(1-e^{-\frac{\hbar\omega}{k_BT}}\right)}$$

2 (c)

$$\omega = 8\times10^{14}\ s^{-1}$$
$$\hbar = 1.05\times10^{-34}\ Js$$
$$k_B = 1.38\times10^{-23}\ J/K$$

We want $P(n=0)=\frac{1}{2}$ so

$$P(n) = (1-e^{-\frac{\hbar\omega}{k_BT}})e^{-n\frac{\hbar\omega}{k_BT}}$$
$$\text{we want } \frac{1}{2} = P(n=0)$$
$$\frac{1}{2} = (1-e^{-\frac{\hbar\omega}{k_BT}})$$
$$\frac{1}{2} = e^{-\frac{\hbar\omega}{k_BT}}$$
$$\ln(\frac{1}{2}) = \ln(e^{-\frac{\hbar\omega}{k_BT}})$$
$$-0.693 = -\frac{\hbar\omega}{k_BT}$$
$$\Rightarrow T = \frac{\hbar\omega}{k_B(0.693)}$$
$$T = \frac{(1.05\times10^{-34})(8\times10^{14})}{(1.38\times10^{-23})(0.693)}$$
$$T = 8783\ K$$

which is very hot indeed.

StatMech 3

3 (a)

Here we have what is called a “joint” probability density. This just means $w(x,v)dxdv$ is the probability of finding the oscillator to be between x and $x+dx$ **and at the same time** having velocity (not speed) between v and $v+dv$. Since the probabilities relating to x and the probabilities relating to v are independent, that means that we can write

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,v)\ dxdv &= 1 \\
\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ce^{-\frac{kx^2+mv^2}{2k_BT}}\ dxdv &= 1 \\
\int_{-\infty}^{\infty} C_x e^{-\frac{kx^2}{2k_BT}}\ dx \int_{-\infty}^{\infty} C_v e^{-\frac{mv^2}{2k_BT}}\ dv &= 1
\end{aligned}
$$

$$\text{individually we have: } \int_{-\infty}^{\infty} C_x e^{-\frac{kx^2}{2k_BT}}\ dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} C_v e^{-\frac{mv^2}{2k_BT}}\ dv = 1$$

Let us find each of these constants individually

$$
\begin{aligned}
\int_{-\infty}^{\infty} C_x e^{-\frac{kx^2}{2k_BT}}\ dx &= 1 \\
\int_{-\infty}^{\infty} C_x e^{-a^2x^2} &= 1 \qquad a = \sqrt{\frac{k}{2k_BT}} \\
C_x \frac{\sqrt{\pi}}{a} &= 1 \\
C_x &= \sqrt{\frac{k}{2\pi k_BT}}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} C_v e^{-\frac{mv^2}{2k_BT}}\ dx &= 1 \\
\int_{-\infty}^{\infty} C_v e^{-b^2v^2} &= 1 \qquad b = \sqrt{\frac{m}{2k_BT}} \\
C_v \frac{\sqrt{\pi}}{b} &= 1 \\
C_v &= \sqrt{\frac{m}{2\pi k_BT}}
\end{aligned}
$$

Putting everything together,

$$
\begin{aligned}
w(x,v) &= Ce^{-\frac{kx^2+mv^2}{2k_BT}} \\
w(x,v) &= C_xC_v e^{-\frac{kx^2+mv^2}{2k_BT}} \\
w(x,v) &= \sqrt{\frac{k}{2\pi k_BT}}\sqrt{\frac{m}{2\pi k_BT}}\ e^{-\frac{kx^2+mv^2}{2k_BT}}
\end{aligned}
$$

3 (b)

$$\begin{aligned}
\langle E \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,v)w(x,v)\ dxdv \\
\langle E \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,v)C_x C_v e^{-\frac{kx^2}{2k_BT}} e^{-\frac{mv^2}{2k_BT}}\ dxdv \\
\langle E \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\frac{1}{2}kx^2 + \frac{1}{2}mv^2) C_x C_v e^{-\frac{kx^2}{2k_BT}} e^{-\frac{mv^2}{2k_BT}}\ dxdv \\
\langle E \rangle &= \int_{-\infty}^{\infty} \frac{1}{2}kx^2 C_x e^{-\frac{kx^2}{2k_BT}}\ dx \underbrace{\int_{-\infty}^{\infty} C_v e^{-\frac{mv^2}{2k_BT}}\ dv}_{=1} + \underbrace{\int_{-\infty}^{\infty} C_x e^{-\frac{kx^2}{2k_BT}}\ dx}_{=1} \int_{-\infty}^{\infty} \frac{1}{2}mv^2 C_v e^{-\frac{mv^2}{2k_BT}}\ dv \\
\langle E \rangle &= \int_{-\infty}^{\infty} \frac{1}{2}kx^2 C_x e^{-\frac{kx^2}{2k_BT}}\ dx + \int_{-\infty}^{\infty} \frac{1}{2}mv^2 C_v e^{-\frac{mv^2}{2k_BT}}\ dv \\
\langle E \rangle &= \frac{kC_x}{2}\int_{-\infty}^{\infty} x^2 e^{-a^2x^2}\ dx + \frac{mC_v}{2}\int_{-\infty}^{\infty} v^2 e^{-b^2v^2}\ dv \qquad \left(a = \sqrt{\frac{k}{2k_BT}}, b = \sqrt{\frac{m}{2k_BT}}\right)
\end{aligned}$$

Use

$$\int_{-\infty}^{\infty} x^2 e^{-a^2x^2}\ dx = \frac{\sqrt{\pi}}{2a^3} \quad \text{and} \quad \int_{-\infty}^{\infty} v^2 e^{-b^2v^2}\ dv = \frac{\sqrt{\pi}}{2b^3}$$

$$\langle E \rangle = \frac{kC_x}{2}\left(\frac{\sqrt{\pi}}{2a^3}\right) + \frac{mC_v}{2}\left(\frac{\sqrt{\pi}}{2b^3}\right)$$

where: $a = \sqrt{\frac{k}{2k_BT}}, \quad b = \sqrt{\frac{m}{2k_BT}}, \quad C_x = \sqrt{\frac{k}{2\pi k_BT}}, \quad C_v = \sqrt{\frac{m}{2\pi k_BT}}$

Now start plugging in expressions for the various constants a, b, C_x and C_v, and tidy up

$$\langle E \rangle = \frac{k}{2}\left(\sqrt{\frac{k}{2\pi k_BT}}\right)\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{2k_BT}{k}\right)^{\frac{3}{2}} + \frac{m}{2}\left(\sqrt{\frac{m}{2\pi k_BT}}\right)\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{2k_BT}{m}\right)^{\frac{3}{2}}$$

After lots of careful cancelation, you get

$$\begin{aligned}
\langle E \rangle &= k_BT\left(\frac{1}{2} + \frac{1}{2}\right) \\
\langle E \rangle &= k_BT
\end{aligned}$$

which is exactly what the equipartition of energy theorem predicts for a system with $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$, i.e. two quadratic terms.