1. Fluid is drained from a cylindrical tank by means of a hole at the bottom of the tank. The cross sectional area of the tank is A, and that of the hole is a, with $a \ll A$. Assume the fluid is lossless and flows with laminar flow, so you can apply Bernoulli's equation.

a. (5 points) Show that the fluid leaves the hole with speed $v = \sqrt{2g\,h(t)}$.

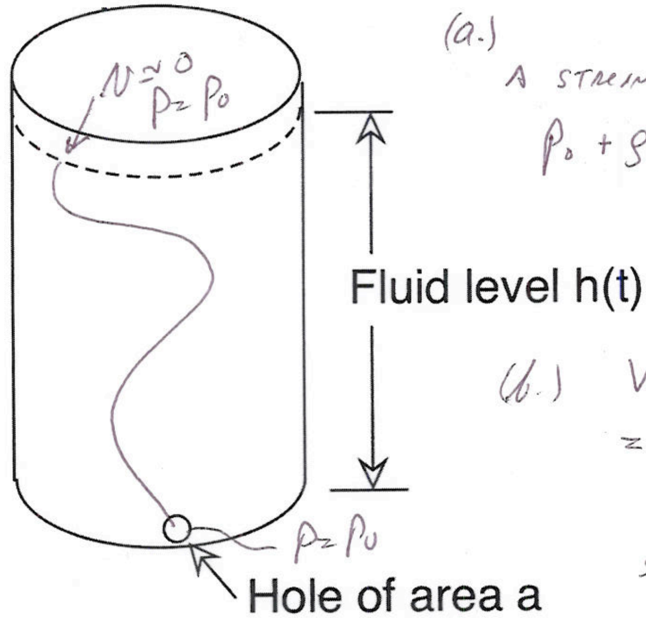
b ~~a~~. (10 points) Derive the differential equation relating the time-dependent fluid height $h(t)$ and its time derivative, and show that it can be written as:

$$-\sqrt{2g}\,\frac{a}{A}\,dt = \frac{dh}{\sqrt{h(t)}}.$$

c ~~b~~. (5 points) Integrate to relate the total time required to drain the tank to the initial fluid level H_o, and show that the time to drain is equal to the time for an object to fall a distance equal to H_o multiplied by the ratio A/a.

Bernoulli $\Rightarrow P + \rho g z + \frac{1}{2}\rho v^2 = \text{CONST.}$

Tank with area A



(a.) START AT SURFACE & FOLLOW A STREAMLINE to the hole

$$P_o + \rho g h(t) + \tfrac{1}{2}\rho(0)^2 = P_o + \rho g \cdot 0 + \tfrac{1}{2}\rho v^2$$

$$\therefore \rho g h(t) = \tfrac{1}{2}\rho v^2$$

$$v^2 = 2gh(t) \qquad v = \sqrt{2gh(t)}$$

(b.) Volume flow rate $= Q = av$

$$= -A\frac{dh}{dt} = a\sqrt{2gh(t)}$$

so $$\frac{dh}{\sqrt{h(t)}} = -\frac{a}{A}\sqrt{2g}\,dt$$ ✓

c. $$\int_{H_o}^{0}\frac{dh}{\sqrt{h}} = -\frac{a}{A}\sqrt{2g}\int_0^{T_D} dt = -\frac{a}{A}\sqrt{2g}\,T_D$$

$$2\sqrt{h}\Big|_{H_o}^{0} = -2\sqrt{H_o} = -\frac{a}{A}\sqrt{2g}\,T_D \qquad \boxed{T_D = 2\frac{A}{a}\sqrt{\frac{H_o}{2g}}}$$

To fall $H_o = \frac{1}{2}gT_F^2 \Rightarrow T_F = \sqrt{\frac{2H_o}{g}} = \frac{A}{a}T_D$ ✓

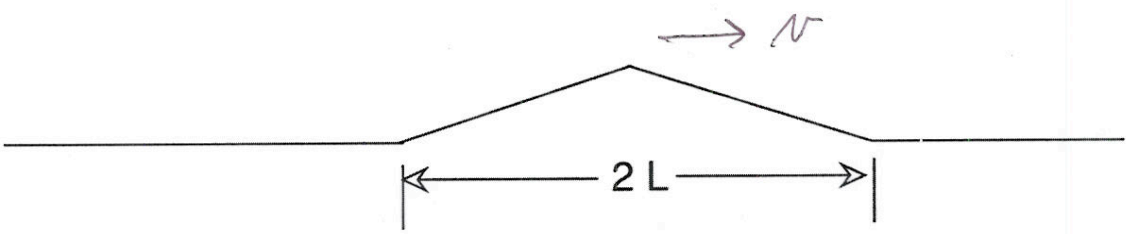
2. A symmetric triangular pulse travels to the right on a string under tension T = 100 N, and having mass per unit length $\mu = 0.01$ kg/m. The pulse has amplitude $Y_o = 0.05$ m and length $2L = 0.5$ m, as shown.

a. (5 points) With what speed does the pulse travel along the string?

b. (10 points) With what speed is the string on the leading edge of the pulse moving up, transverse to the string?

c. (5 points) How much kinetic energy does the pulse contain?

a. $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{100\ N\ m}{0.01\ kg}} = 100\sqrt{\frac{kg\ m^2}{s^2\ kg}} = 100\ m/s$



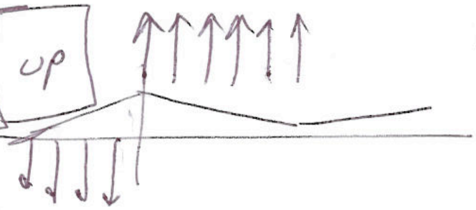
b. This is a wave, so it has the equation $y(x,t) = f(x - vt)$. so $\frac{\partial y}{\partial t} = f' \cdot (-v)$ by the chain rule

$\frac{\partial f}{\partial x} = f'$, so $\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$. On the leading

edge $\frac{\partial y}{\partial x} = \text{const} = -\frac{y_0}{L}$, so $\frac{\partial y}{\partial t} = \frac{v y_0}{L} = 100\frac{m}{s} \times \frac{0.05 m}{0.25 m}$

$\boxed{v_{trans} = \frac{\partial y}{\partial t} = 20\ m/s}$ $\boxed{\text{up}}$

Back is going DOWN



c. $K = \frac{1}{2} m v^2 = \frac{1}{2}(\mu L)\left(\frac{v y_0}{L}\right)^2$ (front) $+ \frac{1}{2}(\mu L)\left(\frac{y_0 v}{L}\right)^2$ (back)

$K = \frac{\mu L y_0^2 v^2}{L^2} = 0.01\frac{kg}{m}(0.05 m)^2 \times \frac{(100 m/s)^2}{0.25 m}$

$= \frac{(0.05)^2 \times 100}{0.25} kg \frac{m^2}{s^2} = \frac{5 \times 0.05}{0.25} J = \boxed{1 J = K}$

3. A sound wave propagates in a medium having a density of 4.0 kg/m^3, and a bulk modulus of 4 x10^4 N/m^2.

a. (5 points) What is the speed of sound in the medium?

b. (5 points) If the pressure at the peak of a wave is 10 N/m^2 above the average pressure, find the amount by which the density exceeds the average at the peak of the wave.

c. (10 points) How many Joules will this sound wave deliver through a 10 cm by 10 cm opening over a period of a minute, assuming it is a sinusoidal wave and the opening is perpendicular to the direction of propagation?

a. $v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{4\times10^4\ N/m^2}{4\ kg/m^3}} = 100\sqrt{\frac{1\cdot kg\ m\ m^3}{s^2 m^2\ kg}} = \boxed{100\ m/s}$

b. $\frac{\delta P}{B} = \frac{\delta\rho}{\rho} \Rightarrow \delta\rho = \rho\frac{\delta P}{B} = 4\frac{kg}{m^3}\times\frac{10\ N/m^2}{4\times10^4\ N/m^2}$

$\boxed{\delta\rho = 10^{-3}\ kg/m^3}$

c. $\bar{I} = \frac{\delta P^2}{2\rho v} = \frac{\bar{P}}{A} = \frac{(10\ N/m^2)^2}{8\frac{kg}{m^3}\times 100\ m/s} = \frac{1}{8}\frac{N^2\ m^2 s}{m^4\ kg}$.

$\bar{I} = \frac{1}{8}\left(\frac{N\cdot m}{s}\right)\frac{N\ m\ s^2}{m^4\ kg} = \frac{1}{8}W\left(kg\frac{m}{s^2}\frac{m\ s^2}{m^4\ kg}\right) = \frac{1}{8}\frac{W}{m^2}$ ✓

so with $A = 0.1\ m \times 0.1\ m = 0.01\ m^2$

and $T = 60\ s$, the energy will be

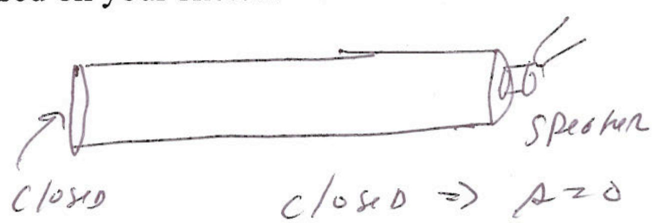
$E = \bar{I}\cdot A\cdot T = \frac{1}{8}\frac{W}{m^2}\times 0.01\ m^2\times 60\ s = \frac{0.6}{8}\ J = 0.075\ J$

$\boxed{E = \frac{0.6}{8}\ J = 0.075\ J}$

4. Consider a tube that is closed at one end and driven by a loudspeaker at the other end. The tube contains a gas in which sound propagates at 300 m/s.

a. (10 points) How long should the tube be in order for the lowest resonant frequency to be 100 Hz? Show by a sketch what the standing wave in the pressure looks like.

b. (10 points) What would the third resonant frequency be in that case? Show by a sketch what the standing wave in the displacement (NOT the pressure) looks like for this situation. Be sure to indicate which end is closed on your sketch.

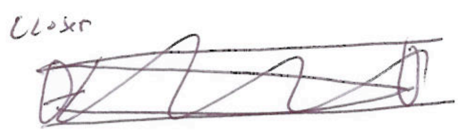


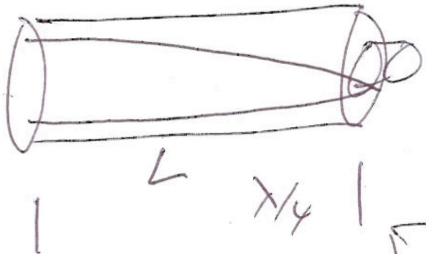
Closed ⇒ $s = 0$ ∴ Displacement node

⇒ Pressure anti-node

Speaker ⇒ Displacement is max, i.e. s anti-node

⇒ Pressure node



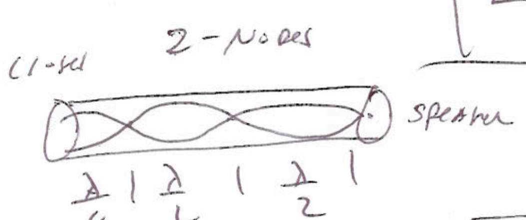


∴ $L = \frac{\lambda}{4}$ $\lambda = 4L$

$\nu_0 = \frac{v}{\lambda} = \frac{v}{4L} = 100\ Hz$

$$\boxed{L = \frac{300\ m/s}{400\ s^{-1}} = \frac{3}{4}\ m = 75\ cm}$$

b.

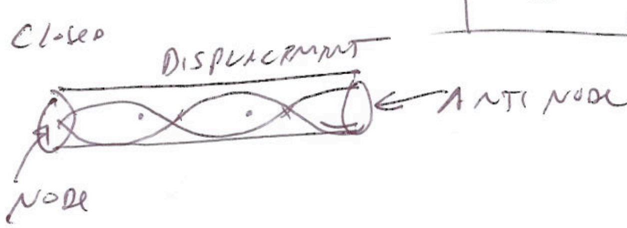


Pressure

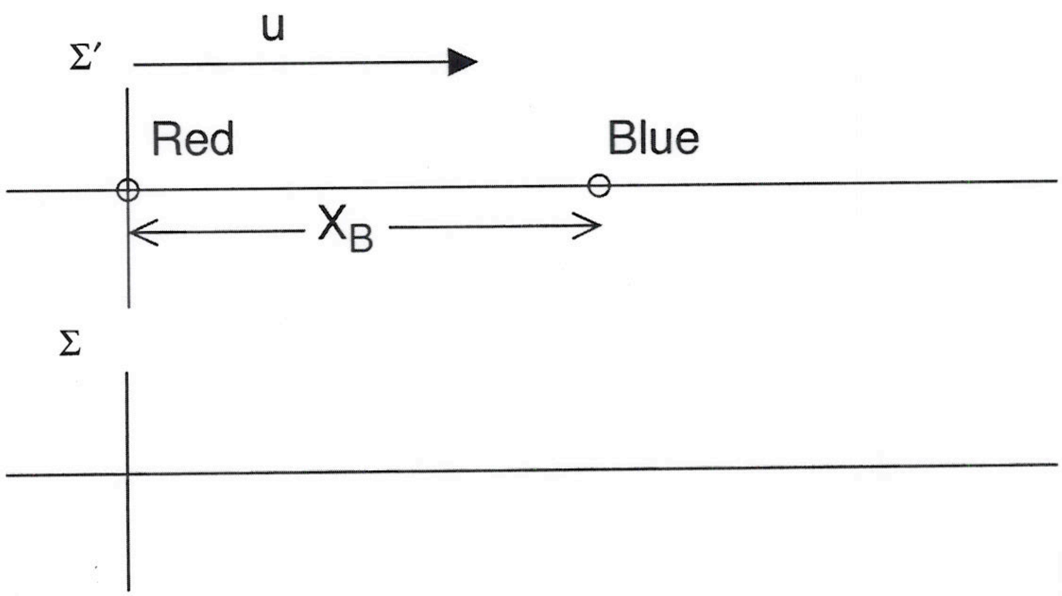
⇒ $\frac{5\lambda}{4} = L$, or $\lambda = \frac{4L}{5}$

$$\boxed{\nu_2 = \frac{v}{\lambda} = 5\frac{v}{4L} = 500\ Hz}$$

s



5. Consider two reference frames, one called Σ and the other called Σ' that is moving toward positive x with speed u, relative to Σ. Call $t = t' = 0$ the time at which the two origins of coordinates cross each other. (In other words, the usual situation.) There are two light emitting diodes (LED's) in the Σ' frame, a red one at the origin, and a blue one located on the x'-axis at $x' = X_B$. As the two origins pass each other, the red LED is flashed. This generates a light pulse that travels out and reaches the blue LED some time later. The arrival of the red pulse immediately triggers the blue LED to flash.



a. (10 points) Show that the time at which the blue LED flashes as measured by an observer in the Σ frame, i.e. the laboratory frame, is equal to
$$\frac{X_B}{c}\sqrt{\left(1+\frac{u}{c}\right)/\left(1-\frac{u}{c}\right)}.$$

IN Σ' it flashes at $t'_B = X_B/c$, $x' = X_B$

$\therefore$ IN Σ $t_B = \gamma\left(\frac{X_B}{c} + \frac{u}{c^2}X_B\right) = \gamma\frac{X_B}{c}(1+u/c)$

$$t_B' = \frac{X_B/c(1+u/c)}{\sqrt{1-u^2/c^2}} = \frac{(X_B/c)(1+u/c)}{\sqrt{(1+u/c)(1-u/c)}} = \frac{X_B}{c}\sqrt{(1+u/c)/(1-u/c)}$$

b. (10 points) Find the position of the blue LED as measured by an observer in the Σ frame, i.e. the laboratory frame, when the light from the blue LED reaches the red one. IN Σ' $t'_R = 2X_B/c$, $x'_B = X_B$

$\therefore$ IN Σ $x_B = \gamma(x'_B + u t'_R) = \gamma(X_B + u\,2X_B/c) = \gamma X_B\left(1+\frac{2u}{c}\right)$

$$\boxed{x_B = \frac{X_B\left[1+2u/c\right]}{\sqrt{1-u^2/c^2}}}$$