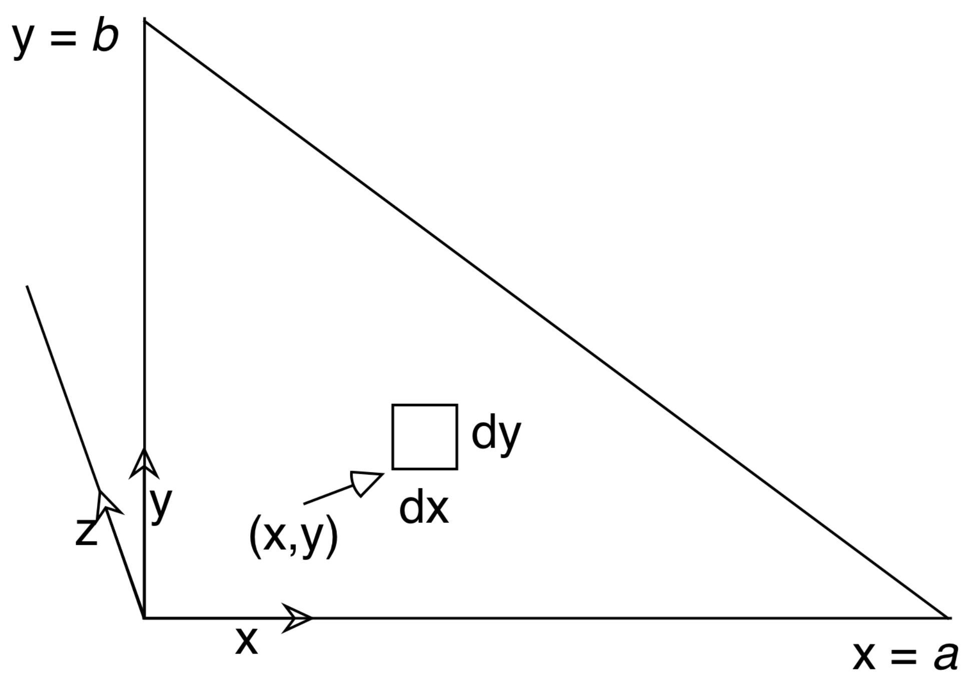
Double Integral Example

Consider a triangular plate of material having a constant uniform density ρ_O, but with a thickness that varies with position, and is given by $t(x,y) = t_O + t_1(x/a)(y/b)$, where a is the dimension of the plate in the x-direction, and b is the dimension in the y-direction, as shown. The constant t_O is the thickness at the origin, and t_1 is a constant that tells us how much thicker is gets as x and y increase. Note that I have been careful to write the equation for the thickness is such a way that t_1 has the units of thickness. This will be useful in revealing how significant the effects of non-uniform thickness are in comparison to a uniform plate.

We want to calculate the mass of the plate and its moment of inertia for rotation about the z-axis, which comes straight up out of the page. Assume the bottom of the plate is flat and lies in the plane of the page, i.e. the x-y plane.



To get the mass, or anything else for that matter, we need to break the plate into small pieces and use integrals to add up the results. We must choose pieces that fill the plate, and set up what we want for a general case. The obvious choice is to use Cartesian co-ordinates, and small square pieces, as shown. In general, such a small piece will be located at a position (x,y), which you may think of as being anywhere you like within the small piece, but I will draw it as being at the lower left-hand corner. The dimension of the little piece in the x-direction is dx, while that in the y-direction is dy. The steps to follow are:

1. Figure out from its dimensions and what we know about density, thickness, etc., whatever property we are interested in for our small piece, e.g. its mass and contribution to the moment of inertia, or center-of-mass position, and write this down explicitly.

2. Write down two integral signs, one inside the other, standing for the process of adding up all the contributions from all the little pieces. Include two differentials, one for the inner integral and one for the outer one.

3. Decide on the proper limits for the two integrals, sticking to the convention that the inner integral goes with whichever differential (dx or dy) is the inner one, and must be done first.

The limits must be set in such a way as to exactly cover the object. In doing the inner integral, the value of the second variable is held constant. This means that we are adding up pieces along a line of constant value for the outer variable. (For example, we have x and the inner variable, then in doing the inner integral, we are adding up all the pieces along a horizontal line.

The limits for the first integral generally depend on the value of the second variable, but the second set of limits must be set at the limits of the object.

4. Carry out the resulting integrals, doing the inner one first, and then integrating that result as the second integral.

For our case, we need to figure out the mass of our little piece to begin. This is actually quite easy. It is so small that the thickness is essentially uniform over its dimension, so its volume is $dV = t(x,y)\,dx\,dy$, and so its mass must be $dM = \rho_O\, t(x,y)\,dx\,dy$. Now we have finished step 1.

For step 2, we get, $M = \iint \rho_O\, t(x,y)\,dx\,dy$, and are then faced with figuring out the proper limits.

The inner integral is over x, so we must think of integrating along a line of constant y value, i.e. a horizontal line. Clearly we must start integrating at x = 0, and stop when we get to the edge of the triangle. How far over we should go clearly depends on the value of y. For example, with y = 0, we would go all the way to x = *a*, but for y = *b*, we should stop at x = 0. In general we need to stop at the edge, for which the equation is y = *b* – (*b*/*a*) x. (Check: x = 0, gives y = *b*, and x = *a*, gives y = 0.) However, we need to know the values of x lying on the edge as a function of y, so we must invert to get x = *a*- (*a*/*b*) y. (Check that you agree before proceeding.) This then, is the upper limit for the inner integral and we can write down the integrals with their proper limits now.

$$M = \int_{y=0}^{y=b} \left[\int_{x=0}^{x=a-(a/b)y} \rho_O \left(t_O + t_1 (x/a)(y/b) \right) dx \right] dy$$

If I have done the inner integral correctly you should get,

$$M = \int_{y=0}^{y=b} \left[\rho_O t_O \left(a - (a/b)y \right) + \rho_O t_1 \frac{y}{ab} \frac{\left(a - (a/b)y \right)^2}{2} \right] dy.$$

Then just plow ahead and do the second integral and you have the mass.

To calculate the moment of inertia, we want to add up each little piece of mass, multiplied by the square of its distance from the z-axis. To do so, we must follow the same procedure and can write down the result directly from our result for M, just inserting $(x^2 + y^2)$ for the square of the distance. This gives us

$$I_z = \int_{y=0}^{y=b} \left[\int_{x=0}^{x=a-(a/b)y} \left(x^2 + y^2 \right) \rho_O \left(t_O + t_1 (x/a)(y/b) \right) dx \right] dy.$$

Being able to analyze a physical situation to the point of writing down the correct integrals is critical. Doing the integrals afterwards can range from easy to a chore to a severe challenge, but is not my our main concern in physics 22!