Physics 220: Problem Set 1 - solution

1. **Kardar, Chapter 1, Problem 5.**

   (a) The critical point is determined by choosing $dP/dn = d^2P/dn^2 = 0$. One obtains $n_c = b/c$ and $k_B T_c = b^2/(2c)$. Then plugging back into $P(n)$, one obtains $P_c = b^3/6c^2$ and finally $k_B T_c n_c/P_c = 3$.

   (b) We use
   $$\kappa_T = -V^{-1}\left.\frac{\partial V}{\partial P}\right|_T = \left[n\left.\frac{\partial P}{\partial n}\right|_T\right]^{-1} = \left[n(k_B T - bn + cn^2/2)\right]^{-1}. \tag{1}$$
   Then clearly for $n = n_c$, one has $\kappa_T \sim [n_c k_B (T - T_c)]^{-1}$ near T_c.

   (c) Here we just set $T = T_c$, and notice that $P - P_c = \frac{c}{6}(n - n_c)^3$.

   (d) We know that $\int_{n-}^{n+} dP/n = 0$. Writing $dP = \left.\frac{\partial P}{\partial n}\right|_T dn$, we just integrate
   $$\int_{n-}^{n+} dn \frac{P'(n)}{n} = 0. \tag{2}$$
   Doing the integral, and using $n_{\pm} = n_c(1 \pm \delta)$, one obtains the equation
   $$\delta = \frac{T}{2T_c}\ln\left(\frac{1+\delta}{1-\delta}\right) \approx \frac{T}{T_c}\left(\delta - \frac{1}{3}\delta^3 + \cdots\right). \tag{3}$$
   Then near the critical point, $\delta \approx \sqrt{3|t|}$, with $t = (T - T_c)/T_c < 0$.

   (e) Once again we must solve $p'(v) = p''(v) = 0$. This gives $v_c = 2b$ and $k_B T_c = a/(4b)$. The critical pressure is then $p_c = p(v_c) = a/(4b^2e^2)$. The ratio becomes $p_c v_c/(k_B T_c) = 2/e^2 \approx 0.27$.

   (f) Use $\kappa_T = -V^{-1}\left.\frac{\partial V}{\partial P}\right|_T = -\left[vp'(v)\right]^{-1}$, and evaluate at $v = v_c$. One obtains, near the critical point,
   $$\kappa_T \approx \frac{2b^2e^2}{at} = \frac{1}{2p_c t} = \frac{be^2}{2k_B(T - T_c)}, \tag{4}$$
   with $t = (T - T_c)/T_c$.

   (g) Finally, Taylor expand $p(v) \approx p_c + \frac{1}{6}p'''(v_c)(v - v_c)^3$. One finds
   $$p - p_c = -\frac{a}{48b^5e^2}(v - v_c)^3 = -\frac{p_c}{12b^3}(v - v_c)^3 = -\frac{2p_c}{3}\left(\frac{v}{v_c} - 1\right)^3. \tag{5}$$

2. **Coupled orders:** Two different order parameters (which we will denote m and n) may occur in the same system, for instance magnetism and superconductivity, or ferroelectricity and magnetism. If they are completely distinct, i.e. characterize different broken symmetries, then the free energy for the system takes the generic form
   $$F = \int d^d\mathbf{x}\left[r_1 m^2 + r_2 n^2 + \frac{u_1}{2}m^4 + \frac{u_2}{2}n^4 + vm^2n^2\right]. \tag{6}$$
   Here we neglect gradients and will just perform a saddle point analysis. Assume $u_1, u_2 > 0$ and $v^2 < u_1 u_2$. The parameters r_1 and r_2 may take either sign, and can be tuned independently by varying two external parameters, such as temperature and pressure.

(a) Draw the phase diagram in the $r_1 - r_2$ plane when the order parameters are uncoupled, $v = 0$.

Here we simply have $m \neq 0$ if and only if $r_1 < 0$ and similary $n \neq 0$ if and only if $r_2 < 0$. So the diagram looks like this:

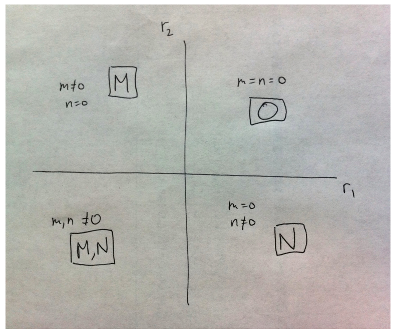

Figure 1: Phase diagram for $v = 0$.

(b) Now consider $v \neq 0$. Find the optimal free energy for the four possible states (local minima) of the system: $m = n = 0$, $m \neq 0, n = 0$, $m = 0, n \neq 0$, and $m, n \neq 0$. What are the conditions on r_1, r_2 such that the latter 3 local minima exist?

- The free energy per unit volume when $m = n = 0$ is $f_{00} = 0$.
- When $n = 0$, we have the usual LG form for m, and so, as in class, $f_{m0} = -r_1^2/(2u_1)$. This exists if $r_1 < 0$.
- $f_{0n} = -r_2^2/(2u_2)$, for $r_2 < 0$.
- To minimize over both m and n simultaneously, it is easiest to just treat $m_2 \equiv m^2$ and $n_2 \equiv n^2$ as variables, in which case F becomes a quadratic function, and is easy to minimize. Differentiating with respect to m_2 and n_2 gives:

$$m_2 = m^2 = \frac{vr_2 - r_1u_2}{u_1u_2 - v^2}, \qquad n_2 = n^2 = \frac{vr_1 - r_2u_1}{u_1u_2 - v^2}. \tag{7}$$

Note that these minima exist only if $m_2, n_2 > 0$. Plugging this back into the free energy gives

$$f_{mn} = -\frac{u_1r_2^2 + u_2r_1^2 - 2vr_1r_2}{2(u_1u_2 - v^2)}. \tag{8}$$

(c) By comparing these four free energies, find the phase diagram for $v > 0$.

For $v > 0$, it is easy to see that when $r_1 > 0$, m must vanish, since all terms in F are monotonically increasing functions of m. Similarly for $r_2 > 0$, n must vanish. Thus in the three quadrants where at least one of r_1 or r_2 is positive, the phase diagram must be unchanged from the $v = 0$ case. In the quadrant with $r_1 < 0$ and $r_2 < 0$, we expect a change, and since $v > 0$, the order parameters "dislike" one another and the region of the (M,N) phase will be decreased. To have both non-zero, we need the numerators in Eq. (7) to be positive (the denominators are always positive by our assumption in the problem set), which then forces $u_1/v < r_1/r_2 < v/u_2$. In this range, one can check that $f_{mn} < f_{m0}, f_{0n}$. So:

(d) Repeat for $v < 0$.

In this case, we expect that the order parameters "like" one another, so that the domain of the (M,N) phase extends outside the lower left quadrant. Consider the case $r_1 < 0$ but $r_2 > 0$.

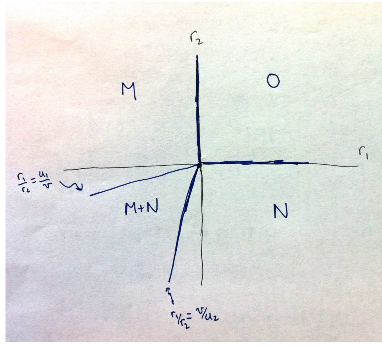

Figure 2: Phase diagram for $v > 0$.

Then definitely $m \neq 0$ but n may vanish. Indeed you can see that m_2 in Eq. (7) is always positive under this condition. The solution for n exists when $r_2 < vr_1/u_1$. Once again when this condition is satisfied, the (M,N) solution has lower energy. So the diagram looks like:

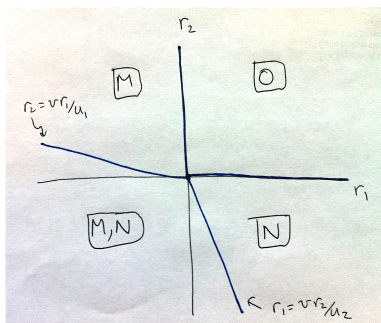

Figure 3: Phase diagram for $v < 0$.

3. **Kardar, Chapter 3, Problem 7.**

(a) We are adding this term to the Landau theory. Transforming to Fourier space, one has

$$\int d^d\mathbf{x} d^d\mathbf{x}' \frac{\vec{m}(\mathbf{x}) \cdot \vec{m}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{d+\sigma}} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} K(q) \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}), \tag{9}$$

where

$$K(q) = \int d^d\mathbf{x} \, \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{|\mathbf{x}|^{d+\sigma}}. \tag{10}$$

We need the small q behavior of $K(q)$. By rotational symmetry, $K(q)$ is a function of q^2 only. We assume $\sigma > 0$, so the integral converges in the limit $q \to 0$. If one naïvely Taylor expands it about $q = 0$, then the first correction would be $O(q^2)$. This diverges for $0 < \sigma < 2$, but converges if $\sigma > 2$. Thus

$$K(q) \approx \begin{cases} K(0) + Aq^\sigma & 0 < \sigma < 2, \\ K(0) + Aq^2 + \cdots + Bq^\sigma & \sigma > 2. \end{cases} \tag{11}$$

There is always a quadratic term so the long-range interaction has no special effect for $\sigma > 2$. It is dominant for $\sigma < 2$.

(b) Bascially we repeat the study in class of the effects of Goldstone fluctuations on the order parameter, but with q^2 replaced by q^σ everywhere. One finds that the phase fluctuations are proportional to

$$\left\langle (\theta(\mathbf{x}) - \theta(0))^2 \right\rangle \sim \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1 - e^{i\mathbf{q}\cdot\mathbf{x}}}{|q|^\sigma}. \tag{12}$$

This diverges as $x \to \infty$ for $d < d_{lc} = \sigma$.

(c) Similarly, we repeat the argument given in class, in which we examined the corrections to the saddle point specific heat. Replacing q^2 by q^σ, we find

$$\delta c \propto \int\int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{(|q|^\sigma + t)^2}, \tag{13}$$

taking for instance $t > 0$. This diverges as $t \to 0$ for $d < d_{uc} = 2\sigma$, in which case it overwhelms the mean field specific heat jump.