Physics 220: Problem Set 2 - solution

1. **Kardar, Chapter 4, Problem 3.**

   (a) The correlation length is determined as $\xi = e^{\ell^*}$, where $\ell^*$ is the scale such that $T(\ell^*)$ is of order one (we are assuming some dimensionless unit here since otherwise the RG equation makes no sense dimensionally). Integrating the RG equation for $T(\ell)$, we obtain

$$\int_T^{T(\ell)} \frac{dT}{T^2} = \frac{n-2}{2\pi} \int_0^\ell d\ell \Rightarrow T(\ell) = \frac{1}{T^{-1} - \frac{n-2}{2\pi}\ell}. \tag{1}$$

   Requiring $T(\ell^*) = 1$ gives $\ell^* = \frac{2\pi}{n-2}(T^{-1} - 1)$, so

$$\xi = e^\ell \sim c e^{\frac{2\pi}{(n-2)T}}, \tag{2}$$

   where $c$ is a constant which is beyond this simple analysis.

   (b) In general, the free energy satisfies the relation

$$f[T, h] = e^{-2\ell} f[T(\ell), h(\ell)]. \tag{3}$$

   Choosing $\ell$ such that $T(\ell)$ is $O(1)$, we obtain

$$f[t, h] = \xi^{-2} g(h\xi^2), \tag{4}$$

   with $\xi$ given in Eq. (2).

   (c) Now we just differentiate twice, so that

$$\chi \sim -\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim \xi^2 f''(0) \sim \xi^2. \tag{5}$$

2. **General recursion relations for an n-component model to one loop:** Consider the very general model Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left\{ \frac{K}{2}|\nabla\vec{m}|^2 + \frac{t}{2}|\vec{m}|^2 + \frac{1}{4!}\sum_{abcd} u_{abcd}\, m_a m_b m_c m_d \right\}, \tag{6}$$

where $a, b, c, d = 1 \cdots n$, and $u_{abcd}$ is a general rank 4 tensor which can be taken to be symmetric under any permutation of $abcd$.

   (a) Derive the differential recursion relation for $u_{abcd}$ to second order in $u$, and leading order in $\epsilon = 4 - d$. We need to integrate out the $\sigma$ fields to second order in $u$, i.e. calculate $\delta(\beta\tilde{\mathcal{H}}) = -\frac{1}{2}\langle \mathcal{U}^2 \rangle_\sigma^c$. Guided by the calculations in class and in Kardar, we know that the only diagrams which will contribute are the *one loop* ones with four external legs. This means in each factor of $\mathcal{U}$, we require the part with two $\tilde{m}$ fields and two $\sigma$ fields. Thus we can write

$$\mathcal{U} = \frac{1}{4!} \times 6 \int d^d\mathbf{x} \sum_{abcd} u_{abcd} \tilde{m}_a \tilde{m}_b \sigma_c \sigma_d + \cdots \tag{7}$$

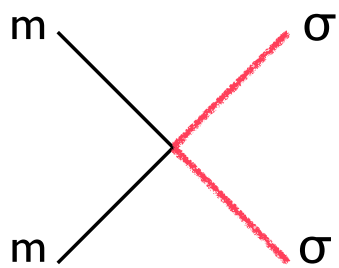


Figure 1: Vertex with two $\tilde{m}$ and two $\sigma$ fields.

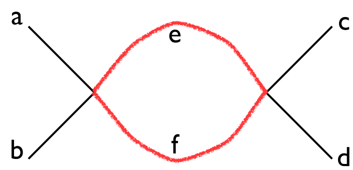


Figure 2: One loop diagram renormalizing the $u$ term.

where the factor of $6$ comes from the ways of choosing which of the four fields to be $\sigma$'s. In diagrams, this corresponds to a vertex with two solid and two wavy lines (Fig. 1). Now we multipy two of these vertices and do the averaging over $\sigma$. There are two (connected) ways of doing the contractions, both of which lead to diagrams looking like Fig. 2. So the correction to the LG Hamiltonian takes the form

$$\delta(\beta\tilde{\mathcal{H}}) = -\frac{1}{2}\left(\frac{1}{4}\right)^2 \times 2I_d \times \sum_{abcdef} \int d^d\mathbf{x}\, u_{abef} u_{efcd}\, \tilde{m}_a \tilde{m}_b \tilde{m}_c \tilde{m}_d, \tag{8}$$

where $I_d = \int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{k}}{(2\pi)^d} 1/(Kk^2+t)^2$. To leading order in $\epsilon$, we can take $I_d \approx \frac{K_4}{K^2} d\ell$. The factors in the above equation are: (1) a $1/2$ from the $-\frac{1}{2}\langle \mathcal{U}^2 \rangle_\sigma^c$ expression, (2) two factors of $1/4 = 6/4!$ from Eq. (7), and (3) a factor of $2$ from the two choices of contractions. Now we need to massage this into the form of the original LG Hamiltonian, in which we have assumed $u_{abcd}$ is symmetric. This requires symmetrizing the tensor in Eq. (8):

$$\delta(\beta\tilde{\mathcal{H}}) = \int d^d\mathbf{x} \sum_{abcd} \left[ -\frac{I_d}{16}\frac{1}{3} \sum_{ef} \left( u_{abef}u_{efcd} + u_{acef}u_{efbd} + u_{adef}u_{efbc} \right) \right] \tilde{m}_a \tilde{m}_b \tilde{m}_c \tilde{m}_d. \tag{9}$$

We can now read off the correction to the $u$ tensor, taking into account the $1/4!$ normalization:

$$\delta u_{abcd} = -\frac{4!}{16 \times 3} \frac{K_4}{K^2} d\ell \times \sum_{ef} \left( u_{abef}u_{efcd} + u_{acef}u_{efbd} + u_{adef}u_{efbc} \right). \tag{10}$$

Repeating the R+R steps from class/Kardar, we finally obtain the differential RG equation

$$\partial_\ell u_{abcd} = \epsilon u_{abcd} - \frac{K_4}{2K^2} \sum_{ef} \left( u_{abef}u_{efcd} + u_{acef}u_{efbd} + u_{adef}u_{efbc} \right). \tag{11}$$

(b) Show that the form of the RG equation for $u$ given in class and in Kardar is recovered if you take $u_{abcd} = 8u(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$.

Now taking the above form of $u_{abcd}$ and plugging in, we need to evaluate three terms which are permutations of

$$u_{abef}u_{efcd} = (\delta_{ab}\delta_{ef} + \delta_{ae}\delta_{bf} + \delta_{af}\delta_{be})\,(\delta_{ef}\delta_{cd} + \delta_{ec}\delta_{fd} + \delta_{ed}\delta_{fc})\,. \tag{12}$$

Sums over $e, f$ are implied. Multiplying this out, we get nine terms, which simplify to

$$u_{abef}u_{efcd} = (n+4)\delta_{ab}\delta_{cd} + 2\delta_{ac}\delta_{bd} + 2\delta_{ad}\delta_{bc}. \tag{13}$$

The other two terms in Eq. (11) can be obtained by permutations of this, so we have in total

$$\partial_\ell u_{abcd} = \epsilon u_{abcd} - \frac{K_4}{2K^2}(8u)^2(n+8)\,(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\,. \tag{14}$$

Happily, the $O(u^2)$ term is now of the same form as the $O(u)$ term. Finally, using the form in the linear terms as well, we get the consistent RG equation for $u$,

$$\partial_\ell u = \epsilon u - \frac{4(n+8)K_4}{K^2}u^2, \tag{15}$$

in agreement with the RG equation for the $O(n)$ model.

(c) Derive the RG equations given in Kardar, chapter 5, problem 4 for the Hamiltonian given in Kardar, chapter 5, problem 3, by using the general equations you derived in part (a) above. The Hamiltonian in Kardar corresponds to taking $u_{abcd} = \hat{u}_{abcd} + 4!\,v\sum_i \delta_{ai}\delta_{bi}\delta_{ci}\delta_{di}$, where $\hat{u}_{abcd} = 8u(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$. Consider then one of the three terms in the RG equation,

$$\begin{aligned} u_{abef}u_{efgh} &= \hat{u}_{abef}\hat{u}_{efgh} + 8\times 4!\,uv(\delta_{ab}\delta_{ef} + \delta_{ae}\delta_{bf} + \delta_{af}\delta_{be})\delta_{ei}\delta_{fi}\delta_{ci}\delta_{di} + (ab \leftrightarrow cd) \\ &\quad + (4!)^2v^2\delta_{ai}\delta_{bi}\delta_{ei}\delta_{fi}\delta_{ej}\delta_{fj}\delta_{cj}\delta_{dj}, \end{aligned} \tag{16}$$

where sums over $i, j, e, f$ are implied when present. Multiplying out the products, one obtains

$$u_{abef}u_{efgh} = \hat{u}_{abef}\hat{u}_{efgh} + 16\times 4!\,uv(\delta_{ab}\delta_{cd} + 2\sum_i \delta_{ai}\delta_{bi}\delta_{ci}\delta_{di}) + (4!)^2v^2\sum_i \delta_{ai}\delta_{bi}\delta_{ci}\delta_{di}. \tag{17}$$

Here the second and third terms in Eq. (16) have been combined into the second term in Eq. (17). The other two terms occuring in the right hand side of Eq. (11) can be deduced by permuting $abcd$ appropriately. One obtains

$$\begin{aligned} &\frac{1}{2}(u_{abef}u_{efcd} + u_{acef}u_{efbd} + u_{adef}u_{efbc}) = \frac{1}{2}(\hat{u}_{abef}\hat{u}_{efcd} + \hat{u}_{acef}\hat{u}_{efbd} + \hat{u}_{adef}\hat{u}_{efbc}) \\ &+ 4!\times 8uv(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + (48uv + 36v^2)\times 4!\sum_i \delta_{ai}\delta_{bi}\delta_{ci}\delta_{di}. \end{aligned} \tag{18}$$

The $\hat{u}$ terms have already been included in part (b). The additional terms are now clearly in the original form and we can identify the renormalized $u$ and $v$ coefficients simply. The result is precisely the RG equations in Kardar, chapter 5, problem 4.

3. **Kardar, Chapter 5, Problem 6:**

(a) Using the standard normalization $m(x_\parallel, x_\perp) = \int \frac{d^n q_\parallel}{(2\pi)^n} \frac{d^{d-n} q_\perp}{(2\pi)^{d-n}} e^{iq_\parallel \cdot x_\parallel + iq_\perp \cdot x_\perp} m(q_\parallel, q_\perp)$, we obtain

$$\beta\mathcal{H}_0 = \int \frac{d^n q_\parallel}{(2\pi)^n} \frac{d^{d-n} q_\perp}{(2\pi)^{d-n}} \frac{Kq_\parallel^2 + Lq_\perp^4 + t}{2} |m(q_\parallel, q_\perp)|^2 - hm(q=0). \tag{19}$$

(b) We let $q_\parallel \to q_\parallel/b$, $q_\perp \to q_\perp/c$, $m \to zm$. Then

$$K' = z^2 b^{-n-2} c^{-(d-n)} K, \qquad L' = z^2 b^{-n} c^{-(d-n+4)} L, \tag{20}$$
$$t' = z^2 b^{-n} c^{-(d-n)} t, \qquad h' = zh. \tag{21}$$

(c) Keeping $K$ and $L$ fixed, we have $z^2 b^{-n-2} c^{-(d-n)} = z^2 b^{-n} c^{-(d-n+4)} = 1$. Dividing the two quantities yields $c^4/b^2 = 1$, so $c = b^{1/2}$. Then we have $z^2 = b^{n+2} c^{d-n} = b^{n+2+(d-n)/2}$. So

$$c = b^{1/2}, \qquad z = b^{(n+d+4)/4}. \tag{22}$$

Then we obtain $t' = b^2 t \equiv b^{y_t} t$ with $y_t = 2$ and $h' = b^{y_h} h$ with $y_h = (n+d)/4 + 1$.

(d) Now we have

$$f(t,h) = b^{-n} c^{-(d-n)} f(t', h') = b^{-(n+(d-n)/2)} f(b^{y_t} t, b^{y_h} h). \tag{23}$$

Letting $b = |t|^{-1/y_t}$, we then obtain the usual form with

$$\alpha = 2 - (n+d)/(2y_t) = 2 - \frac{n+d}{4}, \qquad \Delta = y_h/y_t = \frac{1}{2} + \frac{n+d}{8}. \tag{24}$$

(e) Since $\chi \sim \partial^2 f/\partial h^2$, we obtain

$$\chi \sim |t|^{2-\alpha-2\Delta} \sim |t|^{-1}. \tag{25}$$

(f) This is just a Gaussian expectation value, so

$$\langle m(q) m(q') \rangle_0 = \frac{\delta^{(d)}(q+q')}{Kq_\parallel^2 + Lq_\perp^4 + t}, \qquad \chi_0(q) = \frac{1}{Kq_\parallel^2 + Lq_\perp^4 + t}. \tag{26}$$

(g)

$$U = u \prod_{i=1}^{4} \int \frac{d^n q_{i\parallel}}{(2\pi)^n} \frac{d^{d-n} q_{i\perp}}{(2\pi)^{d-n}} (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3 + q_4) m(q_1) m(q_2) m(q_3) m(q_4). \tag{27}$$

(h) Rescaling according to the above, we obtain

$$u' = z^4 b^{-3n} c^{-3(d-n)} u = b^{(8-d-n)/2} u, \tag{28}$$

so $y_u = (8-d-n)/2$, and hence $d_{uc} = 8-n$.

(i) We have

$$\langle m(q) m(q') \rangle = \langle m(q) m(q') \rangle_0 \tag{29}$$
$$-u \int \frac{d^n q_{i\parallel}}{(2\pi)^n} \frac{d^{d-n} q_{i\perp}}{(2\pi)^{d-n}} (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3 + q_4) \langle m(q) m(q') \times m(q_1) m(q_2) m(q_3) m(q_4) \rangle_0^c.$$

Since all the factors of $m(q_i)$ are symmetric, we can just make any of the 12 possible contractions and multiply by this factor. So

$$\langle m(q)m(q')\rangle = \langle m(q)m(q')\rangle_0 \tag{30}$$

$$-12u \int \frac{d^n q_{i\|}}{(2\pi)^n} \frac{d^{d-n} q_{i\perp}}{(2\pi)^{d-n}} (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3 + q_4) \times$$

$$\langle m(q)m(q_3)\rangle_0 \langle m(q')m(q4)\rangle_0 \langle m(q_1)m(q_2)\rangle_0. \tag{31}$$

(j) Now using the form of these expectation values, and pulling off the $\delta$-functions, we get

$$\chi(q) = \chi_0(q) - 12u\,[\chi_0(q)]^2 \int \frac{d^n k_\|}{(2\pi)^n} \frac{d^{d-n} k_\perp}{(2\pi)^{d-n}} \chi_0(k). \tag{32}$$

As in class, we rearrange this into the series for $1/\chi(q)$,

$$1/\chi(q) = K q_\|^2 + L q_\perp^4 + t + 12u \int \frac{d^n k_\|}{(2\pi)^n} \frac{d^{d-n} k_\perp}{(2\pi)^{d-n}} \frac{1}{K k_\|^2 + L k_\perp^4 + t} \tag{33}$$

Requiring that this vanish at $q = 0$, gives, to leading order in $u$ (where we can neglect the $t$ in the denominator of the integral):

$$t_c = -12u \int \frac{d^n k_\|}{(2\pi)^n} \frac{d^{d-n} k_\perp}{(2\pi)^{d-n}} \frac{1}{K k_\|^2 + L k_\perp^4}. \tag{34}$$