Physics 220: Problem Set 3
due May 19, 2011.

1. **Kardar, Chapter 6, Problem 3.** The solution is in the book and so will not be graded.

2. **Kardar, Chapter 6, Problem 6.**

   (a) We write the Boltzmann weight in the XY model as

   $$e^{-\beta\mathcal{H}} = \prod_i e^{K\vec{s}_i\cdot\vec{s}_{i+1}} = \prod_i e^{K\cos(\theta_i-\theta_{i+1})} \equiv \langle\theta_{i+1}|\hat{T}|\theta_i\rangle. \tag{1}$$

   Therefore $\langle\theta|\hat{T}|\theta'\rangle = e^{K\cos(\theta-\theta')}$. To diagonalize it we must solve

   $$\int_0^{2\pi}\frac{d\theta'}{2\pi}e^{K\cos(\theta-\theta')}f_m(\theta') = \lambda_m f_m(\theta). \tag{2}$$

   Writing $f_m(\theta) = e^{im\theta}$, and changing variables to $\theta' \to \theta - \theta'$, we indeed see that this is an eigenstate with

   $$\lambda_m = \int_0^{2\pi} d\theta' e^{K\cos\theta'}\cos m\theta' = 2\pi I_m(K), \tag{3}$$

   where $I_m(z)$ is the modified Bessel function.

   (b) Up to normalization, the partition function is then $Z = \lambda_0^N$, where N is the number of spins and $\lambda_0 = 2\pi I_0(K)$ is the maximum eigenvalue of $\hat{T}$. Therefore the free energy per site is

   $$\beta f = -\ln Z/N = -\ln\lambda_0 = \text{const} - \ln I_0(K). \tag{4}$$

   At low temperature, $K \gg 1$, and we can approximate the integral in Eq. (3) for $m = 0$ by the saddle point at $\theta' = 0$:

   $$\lambda_0 \approx \int_{-\pi}^{\pi} d\theta e^{K(1-\theta^2/2)} \approx e^K\int_{-\infty}^{\infty} e^{-K\theta^2/2} = \sqrt{\frac{2\pi}{K}}e^K. \tag{5}$$

   Writing $K = J/(k_BT)$, we have

   $$f \approx -J - k_BT\ln\sqrt{\frac{2\pi k_BT}{J}}. \tag{6}$$

   The second term reflects the presence of small fluctuations of the angles of $|\theta| \sim \sqrt{k_BT/J}$ at low temperature. These fluctuations can be arbitrarily small because we treat the spins classically.

   (c) To obtain the correlation length, we need the ratio of the two largest eigenvalues of the transfer matrix,

   $$\xi = -\frac{1}{\ln(\lambda_1/\lambda_0)}. \tag{7}$$

   At low temperature we apply the saddle point approximation to λ_1:

   $$\lambda_1 \approx e^K\int_{-\infty}^{\infty} e^{-K\theta^2/2+im\theta} = \sqrt{\frac{2\pi}{K}}e^K e^{-m^2/(2K)}, \tag{8}$$

   so we find $\xi \sim 2K = 2J/(k_BT)$. Note the power-law divergence of the correlation at low temperature, which is characteristic of 1d systems with continuous symmetry, and very different from what we saw in the 1d Ising chain.

3. **High temperature expansion for the O(n) model on the honeycomb lattice:** Consider the partition function defined by

$$Z = \prod_i \int [d\vec{s}_i] \prod_{\langle ij \rangle} [1 + nt\, \vec{s}_i \cdot \vec{s}_j] \,, \tag{9}$$

where $\vec{s}_i$ are n-component spins of unit length $|\vec{s}_i| = 1$ and $t > 0$. The integrals are defined to be uniform over the n-dimensional sphere, i.e. $\int [d\vec{s}_i] = \int ds_i^1 \cdots ds_i^n \delta(\vec{s}_i \cdot \vec{s}_i - 1)$. Take the sites to reside on a honeycomb lattice, which is the two-dimensional lattice composed of hexagons sharing sides (the links), three of which intersect at each vertex i.

(a) Construct a high temperature expansion for the partition function. What are the diagrams that appear and what is the weight for each diagram?

As usual, we construct the high temperature expansion by multiplying out the factors in the product in Eq. (9) to obtain a sum of terms, and then carrying out the integrals over the $\vec{s}_i$. Each term may be representated by coloring those bonds in which the $nt\, \vec{s}_i \cdot \vec{s}_j$ factor appears. In each such term, a spin may appear zero, one, two, or three times on each site. The cases where the spin appears one or three times vanish by symmetry. Thus the only terms which survive are those in which each site is covered by zero or two bonds. This implies that the colored bonds form closed loops – these are the diagrams. Due to the trivalent nature of the lattice, they are non-intersecting.

Now we need to determine the weight of each diagram. If the spin appears zero times, the integral gives unity. If a spin appears twice, the integral is of the form

$$\int [d\vec{s}_i] s_i^a s_i^b = \frac{\delta^{ab}}{n} \int [d\vec{s}_i] |\vec{s}_i|^2 = \frac{\delta^{ab}}{n}. \tag{10}$$

Here and below we absorb a factor of S_n into the measure of integration, which anyway just appears as a prefactor in the partition function. For each closed loop of k bonds, we have k sites. This comes with an explicit factor of $(nt)^k$ from the expansion of the product, and in addition we obtain a factor of $1/n^k(\delta^{ab}\delta^{bc}\delta^{cd}\cdots\delta^{Qa})$, where $a, b, c, \cdots, Q$ are dummy spin indices for the k sites around the loop. Contracting all the delta functions we are left with $1/n^k\delta^{aa} = n/n^k = n^{1-k}$ for each loop. Thus the weight of each diagram is given by one such factor for each loop, multiplied by the explicit $(nt)^k$ factor for each loop, which gives

$$W = t^{N_{\rm bonds}} n^{N_{\rm loops}}. \tag{11}$$

(b) Construct an expansion for the spin-spin correlation function, $C_{ij} = \langle \vec{s}_i \cdot \vec{s}_j \rangle$. Show that, in the limit of $n \to 0$, this gives just a sum over configurations of a single self-avoiding polymer.

Now we insert two extra factors s_i^a and s_j^a in the partition function in Eq. (9) (and divide by Z). Since we must still have an even number of total factors of spins on each site, we now get (in the numerator) diagrams which contain one *or three* colored bonds ending at sites i and j, and zero or two elsewhere. If there is one bond ending at sites i and j, we have a connected path ending at i and j. One can show that the diagrams with three bonds ending at i or j are negligible as $n \to 0$. To see this, we consider the integral

$$\int [d\vec{s}_i] s_i^a s_i^b s_i^c s_i^d = \frac{1}{n^2 + 2n} \left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta bc \right), \tag{12}$$

which gives the factor associated with such a site. Note that although it involves three bonds, it only scales as $1/n$ for $n \sim 0$. As a consequence, diagrams involving this factor are always proportional to at least one power of n.

Thus we can restrict, in the $n \to 0$ limit, to diagrams involving one connected path from i to j, and in principle some additional loops. All paths are still non-intersecting. The weights for the loops is calculated as above. For the path, we need to recount. If the path contains k links, it has $k+1$ sites including i and j. It therefore comes with a power of $(nt)^k$ from the explicit factor, and a factor of $1/n^{k+1}$ from the integrals in Eq. (10), and one factor of n from the contraction of all the spin indices in the path. Thus the correlation function is given by

$$C_{ij} = \frac{\sum_{\text{path + loops}} t^{N_{\text{bonds}}} n^{N_{\text{loops}}}}{\sum_{\text{loops}} t^{N_{\text{bonds}}} n^{N_{\text{loops}}}}. \tag{13}$$

In the limit $n \to 0$, only diagrams with no loops contribute, and C_{ij} is indeed given by the sum of configurations of a single self-avoiding polymer (i.e. no intersections) weighted by $t^{N_{\text{bonds}}}$.

4. **Kardar, Chapter 7, Problem 11.**

(a) We consider the "change of variables" from s_i to b_{ij}. Is it really one to one? If we fix one spin, say s_1, then all other spins can be deduced from the b_{ij} by adding them up, e.g. $s_2 = s_1 + b_{21}$. More generally, the difference $s_i - s_j$ is determined by a "line sum" from i to j. However, not all choices of b_{ij} are consistent, because we can choose different paths. All paths will be consistent if the loop around one path vanishes. That is, $S_p = b_{ij} + b_{jk} + b_{kl} + b_{li} = 0$, where i, j, k, l are the 4 sites on the plaquette, which is also clearly seen by algebra. So, we can change variables from the full set of s_i to say s_1 and the b_{ij}, provided we fix all $S_p = 0$.
Thus

$$Z = q \sum_{\{b_{ij}\}} \prod_p \delta[S_p]_{\text{modq}} \prod_{\langle ij \rangle} e^{J(|b_{ij}|)}. \tag{14}$$

the factor of q comes from the sum over the spin s_1, and can be neglected in the thermodynamic limit.

(b) Using the representation given in the text, we have

$$\begin{aligned} Z &= q \sum_{\{b_{ij}\}} \sum_{\{n_p\}} \prod_p \frac{1}{q} e^{2\pi i n_p S_p/q} \prod_{\langle ij \rangle} e^{J(|b_{ij}|)} && (15) \\ &= q^{1-N} \sum_{\{b_{ij}\}} \sum_{\{n_p\}} \prod_{\langle ij \rangle} e^{J(|b_{ij}|) + 2\pi i b_{ij}(n_{a(ij)} - n_{b(ij)})/q}, && (16) \end{aligned}$$

where we have grouped all the terms into those associated with one ij bond. Here sites $a(ij)$ and $b(ij)$ are those associated with the two plaquettes bordering the link ij. These share a link $\langle ab \rangle$ of the dual lattice. Now we can carry out the sum over b_{ij} independently on each link to obtain

$$Z = q^{1-N} \sum_{\{n_a\}} \prod_{\langle ab \rangle} \lambda(n_a - n_b), \tag{17}$$

where

$$\lambda(n) = \sum_{b=1}^{q} e^{J(q) + \frac{2\pi i b}{q} n}. \tag{18}$$

In obtaining this we have relabeled ij links by their dual ab counterparts (which cross them), and taken into account $b_{ij} = -b_{ji}$ to obtain the minus sign inside λ.

(c) For the Potts model, we have $J(b) = K\delta^{\rm modq}_{b,0}$. Then

$$\lambda(n) = \sum_{b=1}^{q} e^{K\delta^{\rm modq}_{b,0} + 2\pi i b n/q} = e^K - 1 + q\delta^{\rm modq}_{n,0}. \tag{19}$$

We can also write this as $\lambda(n) = e^{\tilde{K}\delta^{\rm modq}_{n,0} + g}$, to obtain the dual coupling $\tilde{K}$. Equating these two forms, we find

$$e^{\tilde{K}} = \frac{e^K - 1 + q}{e^K - 1}. \tag{20}$$

At the critical point, we expect $\tilde{K} = K$. Solving this equation we obtain

$$K_c = \ln(1 + \sqrt{q}). \tag{21}$$

(d) For the anisotropic Potts model, we consider couplings K_x and K_y on the horizontal and vertical bonds. Each horizontal bond on the direct lattice is associated with a vertical one on the dual lattice (which crosses the direct one), and vice-versa. So we have two relations similar to Eq. (20):

$$e^{\tilde{K}_x} = \frac{e_y^K - 1 + q}{e_y^K - 1} \qquad e^{\tilde{K}_y} = \frac{e_x^K - 1 + q}{e_x^K - 1}. \tag{22}$$

Let us assume that there is one single-valued curve of critical points in the $K_x - K_y$ plane, i.e. $K_x = f(K_y)$. Duality implies that $g(K_y) = f(g(K_x))$ is also critical, where $g(K) = \ln[(e^K + q - 1)/(e^K - 1)]$ is the function appearing in the duality relation. Rewriting this we have $g(K_x) = f^{-1}(g(K_y))$, where f^{-1} is the inverse function to f. Now one can show that $g(g(K)) = K$ (which is why this is a duality!), so acting on both sides of this relation with g gives $K_x = g(f^{-1}(g(K_y)))$. By our assumption that the critical curve is single valued, the right hand side must be equal to $f(K_y)$. Hence we have

$$f(K) = g(f^{-1}(g(K))). \tag{23}$$

This is solved by $f(K) = f^{-1}(K) = g(K)$. (I am not sure how to prove this is unique). But accepting that it is, we see that the critical line is $K_x = g(K_y)$.