# Fluctuations

- So *thermal* fluctuations have a drastic effect in 1d - destroy the phase transition entirely
  - In fact this is a general phenomena: d=1 is the "lower critical dimension" for discrete symmetry breaking at T>0
  - more on this theme later
- What about quantum fluctuation effects at T=0, or thermal fluctuations for d>1?
  - Even when they do not destroy the ordered phase, they alter critical properties and lead to other effects

# Quantum Ising chain

- Coldea $H = \sum_i \left[-J S_i^z S_{i+1}^z - h_\perp S_i^x\right]$
- This can be exactly solved by *Jordan-Wigner transformation*
- First we will reformulate it slightly

$$S_i^z = T_i^x \qquad\qquad S_i^x = -T_i^z$$

$$H = \sum_i \left[-J\, T_i^x T_{i+1}^x + h_\perp T_i^z\right]$$

# Jordan-Wigner

- Idea: spin-1/2 are similar to fermions

$$\{T_i^-, T_i^+\} = 1 \qquad (T_i^+)^2 = (T_i^-)^2 = 0$$

- Transformation

$$T_i^z = \hat{n}_i - 1/2 = c_i^\dagger c_i - 1/2$$

$$T_i^- = U_i c_i \qquad T_i^+ = c_i^\dagger U_i^\dagger$$

$$U_i = e^{i\pi \sum_{j<i} \hat{n}_j} = U_i^\dagger = U_i^{-1}$$

- The "string operator" $U_i$ ensures that spins on different sites *commute*

# Jordan-Wigner

- Exchange term

$$\begin{aligned} T_i^x T_{i+1}^x &= \tfrac{1}{4}(T_i^+ + T_i^-)(T_{i+1}^+ + T_{i+1}^-) \\ &= \tfrac{1}{4}(c_i + c_i^\dagger)U_i U_{i+1}(c_{i+1} + c_{i+1}^\dagger) \\ &= \tfrac{1}{4}(c_i + c_i^\dagger)e^{i\pi\hat{n}_i}(c_{i+1} + c_{i+1}^\dagger) \\ &= \tfrac{1}{4}(c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}) \end{aligned}$$

- Hamiltonian

$$H = \sum_i \left[ -\frac{J}{4}(c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}) + h_\perp(c_i^\dagger c_i - 1/2) \right]$$

**quadratic!**

# Solution

- Fourier $c_j = \frac{1}{\sqrt{L}} \sum_{k \in 2\pi\mathbb{Z}/L} e^{-ikx_j} c_k$
- Hamiltonian

$$H = \sum_k \left[ -\tfrac{J}{4}(c_k^\dagger c_{-k}^\dagger e^{-ik} - c_{-k} c_k e^{-ik} + c_k^\dagger c_k e^{-ik} + c_k^\dagger c_k e^{ik}) + h_\perp c_k^\dagger c_k \right]$$

$$= \sum_{k>0} \left[ -\tfrac{J}{4}(-2i \sin k\, (c_k^\dagger c_{-k}^\dagger - c_{-k} c_k) + 2\cos k\, (c_k^\dagger c_k + c_{-k}^\dagger c_{-k})) + h_\perp (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) \right]$$

- Particle-hole

$$c_{-k} = d_k^\dagger \qquad k > 0$$

# Solution (2)

- Hamiltonian

$$H = \sum_{k>0} \Big[ \frac{iJ}{2} \sin k \, (c_k^\dagger d_k - d_k^\dagger c_k) + (h_\perp - 2J \cos k \,)(c_k^\dagger c_k - d_k^\dagger d_k) \Big]$$

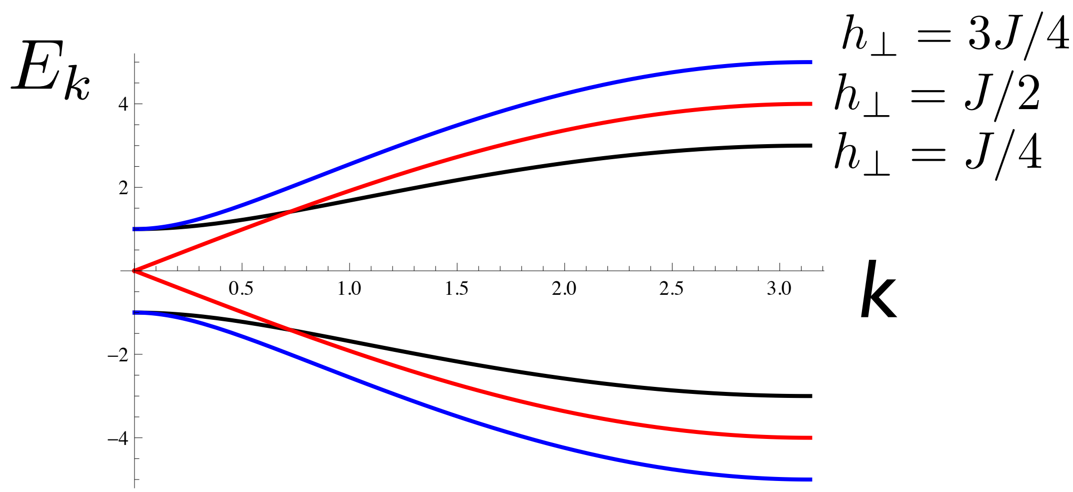
- Spinor $\psi_k = \begin{pmatrix} c_k \\ d_k \end{pmatrix}$

$$H = \sum_{k>0} \psi_k^\dagger \begin{pmatrix} h_\perp - \frac{J}{2} \cos k & i \frac{J}{2} \sin k \\ -i \frac{J}{2} \sin k & -(h_\perp - \frac{J}{2} \cos k) \end{pmatrix} \psi_k$$

# Solution (3)

- Two "bands": $0 < k < \pi$

$$E_k = \pm\left[(h_\perp - \tfrac{J}{2}\cos k)^2 + (\tfrac{J}{2}\sin k)^2\right]^{1/2}$$




- States evolve smoothly *except* at $h_\perp$=J/2, which is qualitatively different: this is the *quantum critical point*

# Phase Diagram