# Phase transition

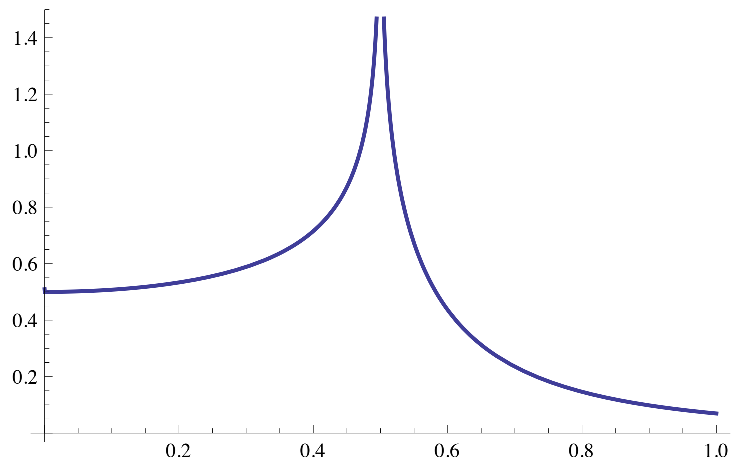
- Ground state energy

$$E = -\sum_k |E_k| = -L\int_0^\pi \frac{dk}{2\pi}|E_k|$$

- Second derivative

$$-\frac{1}{L}\frac{\partial^2 E}{\partial h_\perp^2} = \int_0^\pi \frac{dk}{2\pi}\frac{2J^2\sin^2 k}{(J^2+4h_\perp^2-4h_\perp J\cos k)^{3/2}}$$

"transverse susceptibility" diverges!




this is analogous to specific heat divergence at a classical phase transition

# Correlation Length

- Singularity implies *continuous* transition
  - Can focus on *long-distance* physics

$$E_k = \pm\Big[(h_\perp - \tfrac{J}{2}\cos k)^2 + (\tfrac{J}{2}\sin k)^2\Big]^{1/2}$$
$$\approx \pm\left[(h_\perp - \tfrac{J}{2})^2 + \tfrac{1}{2}h_\perp J k^2\right]^{1/2}$$
$$= \pm\left[\Delta^2 + v^2 k^2\right]^{1/2}$$

$$\Delta = h_\perp - J/2 \qquad\qquad v = \sqrt{h_\perp J/2} \approx J/2$$

$$\xi = v/\Delta = \frac{\sqrt{2h_\perp J}}{2h_\perp - J} \sim (h_\perp - h_\perp^c)^{-\nu} \qquad\qquad \nu = 1$$

# Time scale

- Correlation time scales with ξ

$$\tau \sim \xi/v \sim (h_\perp - h_\perp^c)^{-\nu}$$

- This is consistent with energy-time scaling in quantum mechanics

$$\Delta \sim \hbar/\tau \sim v/\xi$$

- n.b. in general, at a critical point, can have a *dynamical critical exponent z*

$$\tau \sim \xi^z \qquad\qquad z \geq 1$$

# Power laws

- Notice that everything appears to be described by power laws near the QCP
  - This is a general property - "scaling" - of second order phase transitions
- How to understand it?
  - Scale invariance

# Majorana

$$H = \sum_i \left[ -\frac{J}{4}(c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}) + h_\perp (c_i^\dagger c_i - 1/2) \right]$$

- Majorana = real fermions

$$\gamma_j = c_j + c_j^\dagger \qquad \eta_j = i(c_j - c_j^\dagger)$$

- Anticommutators $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$ etc.

$$H = \sum_j \left[ -\tfrac{iJ}{4}\eta_j \gamma_{j+1} + \tfrac{i}{2} h_\perp \eta_j \gamma_j \right]$$

$$\approx \int dx \, \left[ \tfrac{i\Delta}{2}\eta\gamma - \tfrac{iJ}{4}\eta \partial_x \gamma \right]$$

# Majorana magic

- Rotation $\eta = \frac{1}{\sqrt{2}}(\eta_R + \eta_L)$ $\gamma = \frac{1}{\sqrt{2}}(\eta_R - \eta_L)$
- 1d Majorana Hamiltonian

$$H = \int dx \left[ -\frac{iv}{4}(\eta_R \partial_x \eta_R - \eta_L \partial_x \eta_L) + \frac{i\Delta}{2}\eta_L \eta_R \right]$$

critical theory

deviation from criticality

- Δ=0: no intrinsic length scale

$$\eta_{R/L} \sim L^{-1/2}$$

$$H \sim v/L$$

"scaling dimension" of η: $d_\eta = 1/2$

# Effective field theory

$$H = \int dx \, \left[ -\frac{iv}{4} (\eta_R \partial_x \eta_R - \eta_L \partial_x \eta_L) + \frac{i\Delta}{2} \eta_L \eta_R \right]$$

- A critical point is described by a *scale invariant* effective field theory

- *Dimensionless* effective action

$$\mathcal{S} = \int dt dx \, \left\{ \frac{i}{4} \left[ \eta_R (\partial_t - v\partial_x) \eta_R + \eta_L (\partial_t + v\partial_x) \eta_L \right] + \frac{i\Delta}{2} \eta_L \eta_R \right\}$$

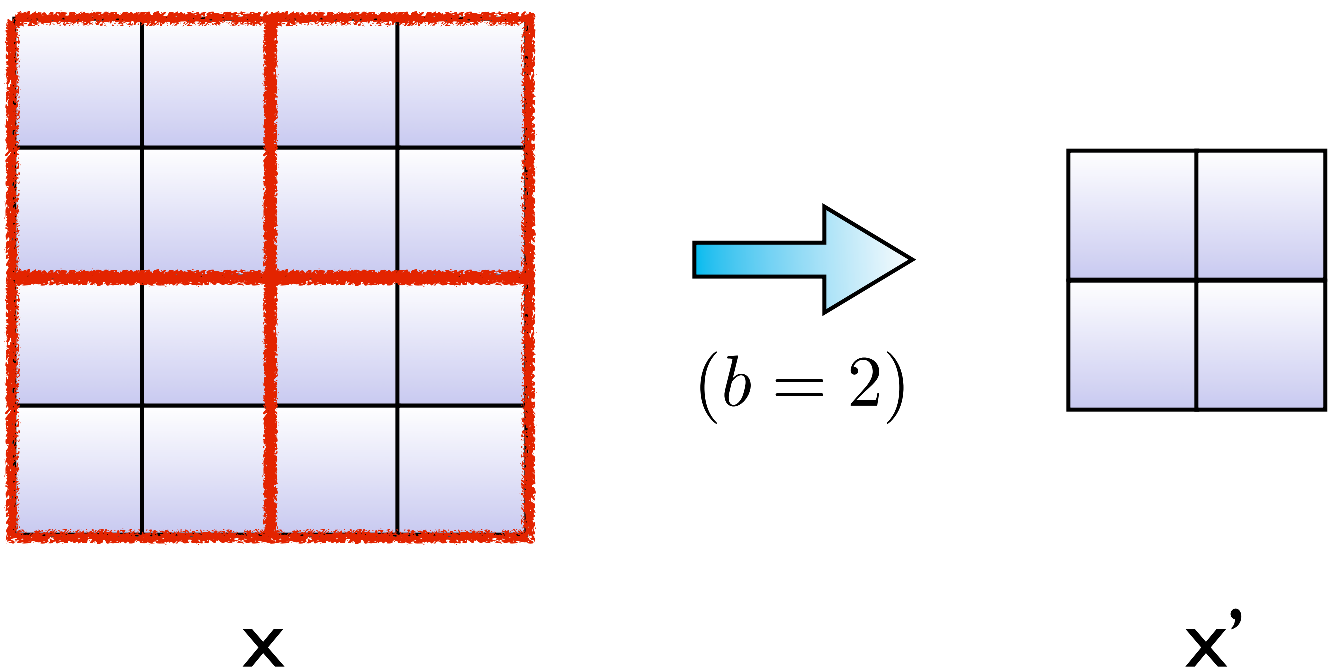
$$t \to b\,t$$
$$x \to b\,x$$
$$\eta_{R/L} \to b^{-1/2}\,\eta_{R/L}$$

critical theory (Δ=0) is invariant under this!

# Scale Invariance

- What does it mean?

$x \to b\,x$ ??    $x = b\,x'$    $b > 1$



# Effective field theory

$$\mathcal{S}_c = \int dt\, dx \, \left\{ \frac{i}{4} \left[ \eta_R(\partial_t - v\partial_x)\eta_R + \eta_L(\partial_r + v\partial_x)\eta_L \right] \right\}$$

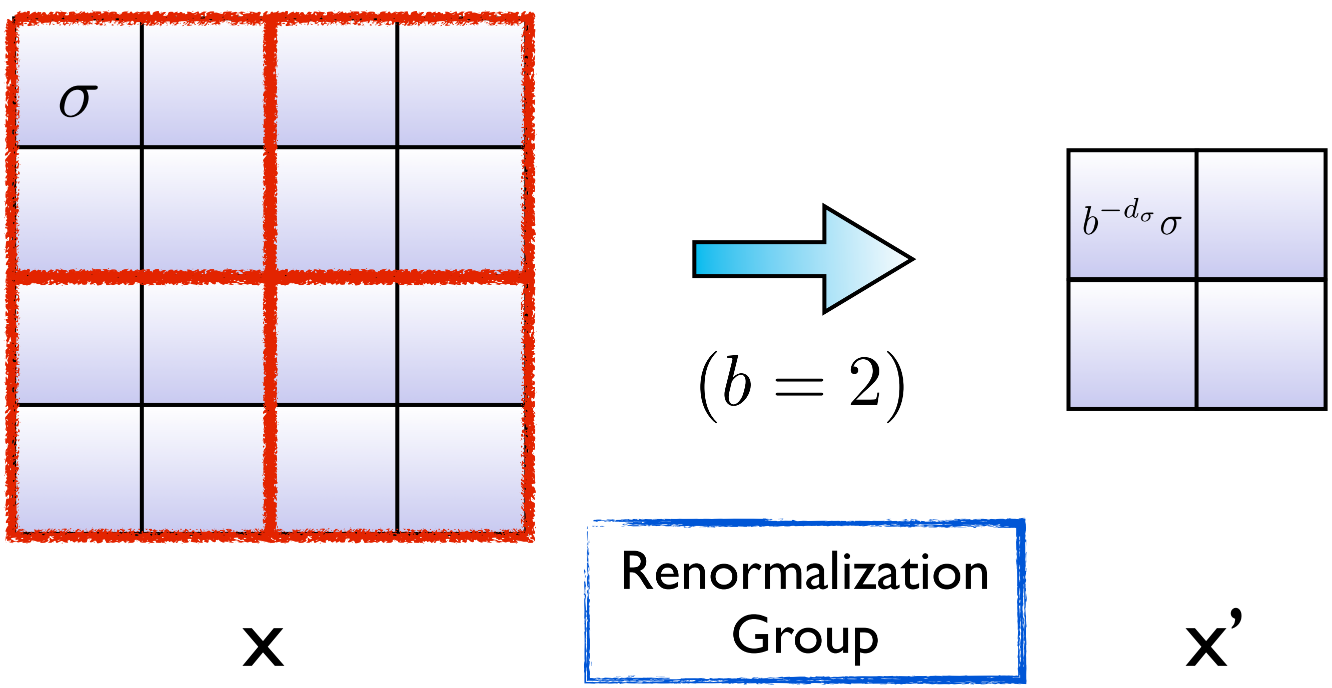
- A critical point is described by a *scale invariant* effective field theory
  - Perturbations are described by *local operators* carrying *scaling dimensions*

| | | |
|---|---|---|
| Fermion | | $d_\eta$ = 1/2 |
| Transverse spin | $\Delta S^x \sim \varepsilon \sim \eta_L \eta_R$ | $d_\varepsilon$ = 1 |
| Ising spin | $S^z \sim \sigma \sim ??$ | $d_\sigma$ = 1/8!! |

# Scale Invariance

- What does it mean?

$x \to b\,x$ ?? $\qquad x = b\,x'$ $\qquad b > 1$



# Renormalization Group

- Perturbations

$$\Delta \mathcal{S} = \int dt\, dx\, \left\{ \Delta h_\perp \, \varepsilon + h_\parallel \, \sigma \right\}$$

- Under RG

$$\Delta h_\perp \to b^{2-d_\varepsilon} h_\perp = b\, h_\perp$$

$$\Delta h_\parallel \to b^{2-d_\sigma} h_\parallel = b^{15/8}\, h_\parallel$$

*relevant* perturbations

- After rescaling, physical quantities with new and old perturbations should be the same