

Physics 220: Problem Set 1 - solution

1. Show by direct calculation that in MFT the susceptibility diverges like  $\chi \sim A/(T - T_c)$  and the specific heat has a jump discontinuity, as claimed in class.

- We repeat the MFT carried out in class, but with an external (infinitesimal) magnetic field  $h$ . The effective field on a spin is then

$$h_{eff} = h + zJm. \quad (1)$$

Then for a free spin,  $m = \chi_0 h_{eff}$ , with  $\chi_0 = A/T$  the Curie susceptibility. Putting this in, we get  $h_{eff} = h/(1 - zJ\chi_0)$ , and hence  $m = \chi_0 h/(1 - zJ\chi_0) = A/(T - T_c)$ , with  $T_c = zJA$ .

- Here we can take zero field. We derived in class that  $m \sim (T_c - T)^{1/2} \Theta(T_c - T)$ . We can just take the expectation value of the energy to obtain the internal energy,  $\langle E \rangle = -J \sum_{\langle ij \rangle} m^2 = -NJzm^2/2$ . This implies  $\langle E \rangle \sim -(T_c - T)\Theta(T_c - T)$ . Taking one derivative,  $C = d\langle E \rangle/dT$ , gives a jump discontinuity.
2. Use the transfer matrix technique to calculate the spin-spin correlation function for the 1d classical Ising model,  $\langle \sigma_i \sigma_j \rangle$ , as a function of  $|i - j|$  and  $K = \beta J$ . Extract the correlation length as a function of  $K$ .

This is an extremely standard calculation, which you can find in many textbooks. For instance, it is in Kardar, p.101-103. The result is  $\xi = -1/\ln(\tanh K)$ .

3. Carry out Curie-Weiss MFT at *zero temperature* for the *quantum* transverse field Ising chain. That is, decouple the exchange term ( $J$  term) to reduce the problem to that of independent spins in both a longitudinal (along  $z$  - this is the effective exchange field) and transverse (along  $x$ ) field. Put each of these spins in its ground state (since it is  $T = 0$ ), and make your solution self-consistent. Find the quantum critical point in mean-field theory and find the longitudinal susceptibility,  $-\frac{1}{L} \frac{\partial^2 E}{\partial h_{\perp}^2}$  (here  $E$  is the ground state energy), in the same approximation.

Note: I am using the convention for the Hamiltonian taken in class,

$$H = -J \sum_n S_n^z S_{n+1}^z - h_{\perp} \sum_n S_n^x, \quad (2)$$

where  $\mathbf{S}_n = \boldsymbol{\sigma}_n/2$  are spin-1/2 spin operators, not Pauli matrices. This is equivalent to

$$H = -\tilde{J} \sum_n \sigma_n^z \sigma_{n+1}^z - \tilde{h}_{\perp} \sum_n \sigma_n^x, \quad (3)$$

with  $\tilde{J} = J/4$  and  $\tilde{h}_{\perp} = h_{\perp}/2$ . So in the latter convention there will be some differences in factors.

- Here we have vector spins. Decoupling, we obtain a vector effective field,

$$\mathbf{h}_{eff} = h_{\perp} \hat{\mathbf{x}} + 2Jm\hat{\mathbf{z}}, \quad (4)$$

where  $m = \langle S_n^z \rangle$  is the Ising magnetization. The MF Hamiltonian is then just  $H = -\sum_n \mathbf{h}_{eff} \cdot \mathbf{S}_n$ .

- The ground state of the MF problem is just the state in which each spin is polarized parallel to  $\mathbf{h}_{eff}$ . Therefore

$$\langle \mathbf{S}_n \rangle = \frac{1}{2} \frac{\mathbf{h}_{eff}}{|\mathbf{h}_{eff}|}. \quad (5)$$

Self-consistency then means

$$m = \frac{Jm}{\sqrt{h_{\perp}^2 + (2Jm)^2}}. \quad (6)$$

This gives the non-zero solution

$$m = 1/2 \sqrt{1 - (h_{\perp}/J)^2}, \quad (7)$$

for  $h_{\perp} < J$ , and  $m = 0$  otherwise. **Thus the critical field in MFT is  $h_{\perp}^c = J$  ( $\tilde{h}_{\perp} = 2\tilde{J}$ ).** Note that MFT overestimates the critical field by a factor of 2 – in class we derived the exact critical field  $h_{\perp}^c = J/2$ .

- To get the longitudinal susceptibility, we need the energy. Following the prior methodology, we note that the ground state energy can be obtained in MFT by taking the expectation of the Hamiltonian,

$$E = \langle H \rangle = -N(Jm^2 + h_{\perp} \langle S_n^x \rangle). \quad (8)$$

Now for  $h_{\perp} < h_{\perp}^c$ ,  $\langle S_n^x \rangle = (h_{eff}^x/h_{eff}^z) \langle S_n^z \rangle = h_{\perp}/(2J)$ . So we have

$$E/N = -Jm^2 - h_{\perp}^2/(2J) = -\frac{J}{4} \left(1 - \left(\frac{h_{\perp}}{J}\right)^2\right) - \frac{h_{\perp}^2}{2J} = -\frac{J}{4} - \frac{h_{\perp}^2}{4J}, \quad (9)$$

for  $h_{\perp} < h_{\perp}^c = J$ . Above the critical field,  $m = 0$  and the spin is just polarized along  $x$ , so

$$E/N = -\frac{h_{\perp}}{2}, \quad (10)$$

for  $h_{\perp} > h_{\perp}^c = J$ . Note that  $E/N$  is continuous at  $h_{\perp} = J$ . Taking the derivatives, we get

$$\chi_l = -\frac{1}{N} \frac{\partial^2 E}{\partial h_{\perp}^2} = \begin{cases} 1/(2J) & h_{\perp} < J \\ 0 & h_{\perp} > J \end{cases}, \quad (11)$$

Much like the specific heat in classical MFT, the longitudinal susceptibility has a jump discontinuity in the quantum model in MFT.