Week 8 [1]

6 Loop calculations and counter-terms

So far, we have seen that we are able to remove the vacuum bubble diagrams without affecting amplitudes. The reason is that we have demanded that

$$\langle 0|0\rangle = 1 \tag{1}$$

This requirement of proper normalization, plus the fact that diagrams formally form an exponential of connected diagrams makes it so that we can ignore various graph contributions without ever having to calculate them explicitly. This is true *even if the contributions look like divergent integrals.*

The other requirement that we will impose is that

$$\langle 0|\phi(x)|0\rangle = 0 \tag{2}$$

Obviously if we have a situation where

$$\langle 0|\phi(x)|0\rangle = f(x) \tag{3}$$

we can use translation invariance of the vacuum to show that $f(x)$ is independent of x. We do this as follows:

$$\langle 0|\partial_\mu\phi(x)|0\rangle \sim \langle 0|[P_\mu, \phi(x)]|0\rangle = \langle 0|P_\mu\phi(x) - \phi(x)P_\mu|0\rangle = 0 \tag{4}$$

since $P_\mu|0\rangle = 0$, and P_μ is selfadjoint. This is $\partial_\mu f = 0$.

Let us call $\langle\phi\rangle(x) = c$. Then if we redefine the field $\phi \to \phi(x) = \tilde\phi(x) + c$, then we find that for $\tilde\phi$ the condition described above is true.

Why do we need this condition? Well, without this condition the LSZ formula does not work. The LSZ formula assumes that$\langle\phi\rangle(x)$ is zero.

The reason why we need to do this shift is that even if $\phi(x) = 0$ in the free field theory, we can have quantum corrections from higher order diagrams that would make the right hand side of equation [3] non-vanishing.

Feynman diagrams that can contribute to the right hand side of 3 are called tadpole diagrams. They are characterized by having a single "external leg" to the graph.

However, we know of another way that we can implement this constraint. We need to have

$$\frac{\delta}{\delta J(x)} W[J]|_{J=0} = 0 \tag{5}$$

This is enough to guarantee that $\frac{\delta}{\delta J(x)} Z[J]|_{J=0} = 0$.

The diagrams that contribute to this equation are of the following graphical form:

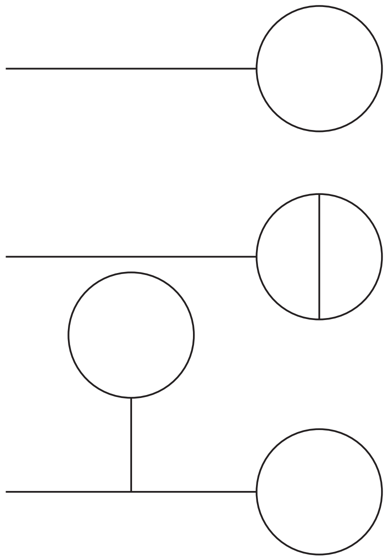

Figure 1: Typical tadpole diagrams: they all have one external leg to the graph (subgraph)

To cancel them, we just need a (single) diagram ending on an x. This is, a vertex where only one line can end.

Figure 2: Counterterm diagram for tadpoles

These diagrams can be absorbed in a coupling constant of the lagrangian

$$\int d^{d+1}x Y \phi(x) \tag{6}$$

Y is zero to zeroth order in perturbation theory, and at higher orders it gets corrections from tadpole diagrams. Once we have decided that Y is non-zero, there will be higher contributions to Y that can involve powers of Y itself.

The key that makes it all hang together is that we can think of Y as follows.

$$Y = \sum_{j \geq 1} \alpha_j g^j \tag{7}$$

as a formal power series, where the goal is to determine the α_j order by order so that equation 3 is true to a given order in a power series in g. When we have higher powers of Y on the right hand side, they come with higher powers of g, so the definition lets us determine Y recursively as a power series in g.

Once we have the term Y in the lagrangian we need to keep it for the future and include it in every single calculation Since Y starts it's life as zero, and we need to introduce it at higher orders to cancel terms that damage one of the requirements for the LSZ formula to be valid, we are using this coupling to keep the physics simple. Terms that begin their life as zero and receive higher order corrections are called counter-terms.

The great thing about this process is that once we know that the tadpoles are cancelled by the Y counter-term Feynman rule, we can use this knowledge in every single graph where we encounter a tadpole connected by a single leg to a more complicated Feynman diagram. The leg connecting the tadpole to the graph is always at zero momentum, so Y cancels a number, not a function of momentum.

This is, Y will not just cancel the tadpole term that is linear in J, but it will cancel every tadpole attached to any graph. Thus, instead of adding more complicated Feynman rules, Y is removing various graphs from consideration.

The other amazing thing about Y is that in most cases Y is formally infinite in value.

This might make you feel uncomfortable: after all, expressions like $\infty - \infty$ can have any value and it would seem that we are cheating when we do these manipulations.

The idea of renormalization is that these expressions are not infinite in practice. Instead, they seem like they are infinite because we are assuming that our theory is valid beyond its regime of validity, for example, that the expressions we find are arbitrarily precise at short distances, beyond the limits where they have been tested experimentally.

The typical (one loop) divergent integral will be of the form bellow, where the divergence comes from regions where $\ell \sim \infty$.

$$\int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{(\ell^2 + \ell \cdot k + B)^m} \sim \int d^{d+1}\ell \ell^{-2m} \sim \Lambda^{d+1-2m} \tag{8}$$

To get the formal finite answer, we cutoff the integral so that all variables have range bounded by Λ. If m is sufficiently large, then the integrals are finite, and we can take Λ to infinity without trouble. If the integral is formally infinite, we can collect the divergent pieces (those with positive powers of Λ. These integrals are also multiplied by coupling constants g^s where s is positive. If we take the point of view that g is very-very small, then the integrals (for finite Λ) are actually very small.

Thus, any counterterm that we add should be technically small in the sense above. If the counter-terms are chosen to balance all the bad Λ behavior, the answer for the integral gets modified by counter-terms that make it well behaved. With this technique, the final answer is finite with good Λ dependence, so that we can take the limit $\Lambda \to \infty$ in the final expression and we get a final answer that is actually finite.

Thus we can work with infinities *as if they are small.* This is because what is really small is the coupling constants themselves, and they multiply the seemingly divergent integrals.