Week 1 [1]

1 Unitary representations of the Poincare group

One of the most important physical concepts that we have available to solve problems in physics is the use of symmetries. For relativistic field theory and quantum mechanics, we need the set of symmetries of the special theory of relativity: Poincare invariance of the dynamics.

At the level of quantum mechanics, symmetry transformations are realized on the Hilbert space of states by unitary transformations.

Thus in order to make sense of relativity together with quantum mechanics, we need to explore in more detail what is necessary for Lorentz invariant to be made consistent with a field theory.

The Poincare group is a continuous group of symmetries, so a lot of the questions that one is required to solve depends on understanding infinitesimal trasnformations of the Poincare group, and it boils down to understanding the set of (unitary) irreducible representations of the Lie algebra of the group in question.

The Lie algebra consists of the following commutation relations:

$$
\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i g_{\nu\rho} P_\mu - i g_{\mu\rho} P_\nu && (1)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= i g_{\nu\rho} M_{\mu\sigma} + i g_{\nu\sigma} M_{\rho\mu} - i g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\rho\mu} && (2)\\
[P_\mu, P_\nu] &= 0 && (3)
\end{aligned}
$$

The M generators are rotations of space-time, while the P are translations.

We are interested in classifying the unitary irreducible representations of the Lorentz group. The unitarity requirement is quantum mechanics, while the irreducibility means that a single state determines completely the action of the special relativity symmetries for all rotations of the given state.

This is a typical problem in abstract quantum mechanics. We have some algebra of operators, and we want to study what constraints follow from that algebra. For example, the angular momentum algebra predicts the quantization of angular momentum in units of $\hbar/2$.

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The typical problem requires us to find a complete set of commuting operators that can serve to characterize all states. For the angular momentum algebra these are usually L^2 and L_z.

This is what we want to do now for the Poincare group.

The first thing we notice is that the P_μ commute with each other. It is therefore convenient to go to a basis where all the P_μ are diagonalized simultaneously.

Given P_μ, it is clear that $P_\mu P^\mu = P^2$ commutes with all generators of the Lorentz group. This is minus the mass square of a particle, for a single particle state. Operators that have this property are called Cassimir operators of the algebra. They serve to classify irreducible representations.

At first sight, it seems that there is no way to make elements of M commute with these conditions. However, if we go to the center of mass frame, where the center of mass is not moving, then in classical physics we have to remember that

$$L_{tot} \sim r_{cm} \times p_{cm} + L_{int} \tag{4}$$

the total angular momentum is a sum of the center of mass angular momentum plus the intrinsic angular momentum. If the center of mass is not moving, then $p_{cm} = 0$, and all angular momentum is intrinsic. This means that if we measure it in any other coordinate system which is stationary with respect to the center of mass, the angular momentum will always point in the same direction, as it does not depend on the position of the center of mass. This position does change between coordinate systems.

It seems that there should be some meaning to angular momentum in the center of mass frame. However, this is not obvious from the commutation relations above.

Again, if we think classically, P_μ is a vector that we can rotate by a Lorentz transformation to find a frame where the center of mass does not move. For this case, $P^\mu \sim (E, 0, \dots)$. The only non-vanishing component of P is the energy. The generators of angular momentum that we need classically are those that rotate space into each other, but that do not involve time. This is, the $M_{\mu\nu}$ with $\mu, \nu \neq 0$.

So long as $E \neq 0$, we can select these by the following operation (in four dimensions):

$$\tilde{M}_\mu \sim \frac{1}{2\sqrt{-P^2}} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \tag{5}$$

The epsilon tensor guarantees that in general M involve only those coordinates that are spacelike and have no time component in the center of mass frame. It is convenient to remove the $\sqrt{-P^2}$ in the denominator and to introduce the Pauli-Lubanski vector, given by

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma \tag{6}$$

It is straightforward to show that (regardless of frame)

$$W_\mu P^\mu = 0 \tag{7}$$

But more importantly, we have that

$$[W^\mu, P_\rho] = 0 \tag{8}$$

Thus W commute with P. However, the W are essentially the angular momentum in the center of mass frame, and they do not commute with each other.
[2]

However, just like in the angular momentum case, we can use $W^2 = W_\mu W^\mu$ as an invariant version of L^2 in the center of mass.

Because all indices are contracted in W^2, this object necessarily commutes with the M. From this, it follows that W^2 is also a Cassimir operator of the Lorentz group.

This is the complete classification of the Cassimir operators of the Poincare group. In higher dimensions $(d+1)$, W is an antisymmetric tensor with $d-2$ indices (the dual tensor to angular momentum in the center of mass frame).

In three dimensions W is a scalar. In 2 dimensions W vanished identically.

In any case, it is easy to see that the the only consistent values of W^2 are

$$W^2 = (-P^2)s(s+1) = M^2 s(s+1) \tag{9}$$

where s is a half-integer.

For a given particle, the value of s appearing above is called the intrinsic spin of the particle.

[2]This idea of selecting the group of transformations that leave the momentum invariant is due to Weyl. This subgroup that has this property is called the *little group*. The representation is built in terms of the so-called method of induced representations. Weinberg's book has avery good discussion about this.

A vector particle has $s = 1$. For the scalar field, a single particle has intrinsic spin $s = 0$. Lorentz vector indices have spin one, and it is easy to show that having many scalar and vector particles can only generate objects of integer spin. This is because of the theory of addition of angular momentum. It's also because if s is a half integer, then rotations by 2π introduce a $(-)$ sign in the wave function. With integer spins, this can never happen.

In nature we observe particles with $s = 1/2$. Thus, in the theory of representations of the Lorentz group, we need to go beyond Lorentz indices to be able to deal with these objects.

This new type of indices that correspond to half-integer spin are called spinor indices, and they are required to make a proper theory of the representations of the Lorentz group.

The quantum fields that carry a single one of these indices are called spinor fields.

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