Physics 223b: Final Exam
You may consult the class notes, Solyom, and previous homeworks and solutions.
Do not discuss the exam problems with anyone except me!
due March 14, 2014 by 4pm, to my office in KITP

1. **IQHE versus TI:** Consider three Hall bars: one made of a two dimensional TI, one made of an IQHE state with filling factor 1, and another made of an IQHE state with filling factor 2. Under ideal conditions, how do their *two terminal* conductances (i.e. with large source and drain electrodes at either end) compare?

2. **Topological insulator without inversion symmetry:** Consider the three-dimensional Dirac Hamiltonian discussed in class,

$$H_0 = v \sum_{\mu=1}^{3} k_\mu \Gamma_\mu + m\Gamma_4. \tag{1}$$

   (a) If inversion symmetry is broken, new terms may be added to the Hamiltonian, i.e. $H \to H_0 + H'$, with

$$H' = \sum_{a>b} c_{a,b} \Gamma_{a,b}. \tag{2}$$

   Assuming $c_{a,b}$ is independent of $\vec{k}$, which $c_{a,b}$ can be non-zero? Assume time-reversal symmetry is maintained.

   (b) Now consider the specific case where only $c_{1,4} = c$ and all other $c_{a,b} = 0$. What is the spectrum? Find the locus of zero energy states when $m = 0$.

3. **Single ion physics:** Consider an Yb^{3+} ion on a site of trigonal symmetry. There is symmetry under a three-fold rotation around the z axis passing through the site, and time-reversal symmetry is preserved. Assume that the ion has no 6s electrons.

   (a) According to Hund's rules for spherical symmetry, what are the S, L and J quantum numbers for the ion?

   (b) Now take into account the crystal field effects, assuming they are weaker than all the physics going into Hund's rules. This means that we can write an effective crystal field Hamiltonian which acts in the ground state J multiplet, in terms of $\vec{J}$ alone. Prove that the most general *quadratic* Hamiltonian describing the spin is just

$$H = c\, J_z^2 + \text{const}. \tag{3}$$

   Sketch the energy levels, indicating their degeneracies, for $c > 0$ and $c < 0$.

4. **Triangular lattice antiferromagnet:** Consider a nearest-neighbor Heisenberg antiferromagnet on the triangular lattice,

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j. \tag{4}$$

(a) Show that the classical ground state consists of a three-sublattice structure, with spins on sublattice A all having $\vec{S}_i = \vec{S}_A$ and similarly for sublattices B and C. Show that $\vec{S}_A$, $\vec{S}_B$ and $\vec{S}_C$ point at 120° angles to one another, and add to zero. Hint: write the Hamiltonian as a sum of squares on each triangle.

(b) Calculate the critical temperature for this state, for the case of $S = 1/2$, in MFT. Here you must assume $\langle \vec{S}_i \rangle = m\hat{S}_A$ when i is on the A sublattice, etc. Compare the T_c you obtain to the critical temperature of the Heisenberg *ferromagnet* on the same lattice with the same spin $S = 1/2$ and the same magnitude of exchange J (but $J \to -J$ in Eq. (4) above).

5. **Magnetic field:** Please estimate the magnitude of the magnetic field in the following two situations. Express your answer in terms of fundamental constants and the basic lengths ξ and/or λ of the superconductor:

   (a) At the center of a superconducting vortex for which all other vortices are much further than λ away.

   (b) At the center of a superconducting vortex in a type II superconductor just below the upper critical field H_{c2}.

6. **Screened superconductivity:** The brilliant Dr. Whackiani finds a way to dramatically screen the charge of electrons, so that effectively, an electron interacts, beyond a few Å, as though it had only one hundredth of its actual charge. When she does this to existing superconductors, do they become more type I or type II?

7. **D-wave in a field:** Consider the two-dimensional d-wave superconductor on the final homework. Suppose a Zeeman magnetic field term is added to the Hamiltonian

$$H_Z = -h \sum_k \left[c^\dagger_{k\uparrow} c_{k\uparrow} - c^\dagger_{k\downarrow} c_{k\downarrow} \right], \tag{5}$$

   with $h > 0$. For $h \ll \Delta, \epsilon_F$, you can assume that the other parameters of the superconducting Hamiltonian are unchanged.

   (a) Are there any zero energy states? If so, where are they in momentum space?

   (b) Sketch the density of states.