Physics 223B: Homework 1 - solutions

1. **Layers:** Suppose n two-dimensional topological insulators are stacked atop one another. Assuming electrons can tunnel between adjacent layers, and disorder is present, for which n can you *guarantee* the existence of a conducting channel at the boundary/edge? Why?

   The answer is that a conducting channel – a helical edge – is guaranteed when n is odd. There are several ways to see this. One is to realize that any finite stack of 2d layers is just another 2d system (one can think of the layers as just orbitals within a unit cell). Consequently one can just consider band inversions at each 2d TRIM. The total inversions just add, so that, since by assumption each layer as an odd number of band inversions, if we add up an odd number of these, we get another odd number, so that the result is still topologically non-trivial. If the number of layers is even, then the Z_2 index will be even, and there is no protection. One can also think of just the edge channels, and realize that one cannot fully gap the odd number of channels without breaking time-reversal.

2. **3d Dirac Symmetries:** In class, we used time-reversal and inversion symmetry to derive the general form for the effective Hamiltonian near a point in which odd and even parity bands exchange. We argued that it took the form of the Dirac Hamiltonian:

$$H = \sum_{\mu=1}^{3} v_\mu k_\mu \Gamma_\mu + m\Gamma_4. \tag{1}$$

   The matrices Γ_a ($a = 1 \cdots 5$), $\Gamma_{a,b} = -\frac{i}{2}[\Gamma_a, \Gamma_b]$ (the brackets indicate commutator, and $a < b = 1 \cdots 5$), and the identity form a set of 16 linearly independent matrices that span the full space of all possible 4×4 matrices. Please determine the transformation properties of all these matrices under time-reversal and inversion symmetries.

   Obviously the identity is invariant under both symmetries. Since the Hamiltonian itself is inversion (P) and time-reversal (T) invariant, we can deduce the transformations from it. Γ_4 enters H directly, and hence must be even under both P and T (in fact, we can regard Γ_4 as the parity operator P). Since k_μ is odd under both P and T, $\Gamma_1, \Gamma_2, \Gamma_3$ are odd under both P and T as well. Now the Γ matrices satisfy $\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 = -1$ *Note: to get this right, you must use the definitions I gave in class for the Euclidean gamma matrices.* This implies Γ_5 must be odd under both P and T as well. This also follows from the identification of Γ_4 with parity. A compact form of this given by defining $\epsilon_4 = 1$ and $\epsilon_a = -1$ for $a \neq 4$. Then under P or T, we have $\Gamma_a \to \epsilon_a \Gamma_a$. These relations can also be determined from the explicit matrices given in class.

   Now we can consider the $\Gamma_{a,b} = -\frac{i}{2}[\Gamma_a, \Gamma_b]$. Under parity, we just get $P : \Gamma_{a,b} \to \epsilon_a \epsilon_b \Gamma_{a,b}$. Under time reversal, which is anti-unitary, there is an extra minus sign due to the complex conjugation: $T : \Gamma_{a,b} \to -\epsilon_a \epsilon_b \Gamma_{a,b}$. Note that the opposite sign under T and P immediately implies that none of the $\Gamma_{a,b}$ are invariant under both P and T.

3. **Weyl semimetal:** Consider the 3d Dirac Hamiltonian with the coordinates rescaled so that the velocity is uniform:

$$H = v \sum_{\mu=1}^{3} k_\mu \Gamma_\mu + m\Gamma_4. \tag{2}$$

Imagine a perturbation is applied of the form

$$H' = b\Gamma_{1,2}. \tag{3}$$

This corresponds to the formation of a certain type of magnetic order with strength proportional to b.

(a) Find the energies of the combined Hamiltonian $H + H'$. I trust you can do this, by hand or by mathematica. I used mathematica. One finds:

$$E_{s,s'} = s\sqrt{m^2 + b^2 + v^2|k|^2 + 2bs'\sqrt{m^2 + v^2k_z^2}}, \tag{4}$$

with $s, s' = \pm 1$ giving four solutions.

(b) What is the condition that there are states at zero energy? Where do those states occur in momentum space (it should correspond to some discrete points $\vec{k} = \vec{K}_i$). Obviously if $E_{+,s'} = 0$ then also $E_{-,s'} = 0$. And this can happen only if $s' = -1$ obviously. Then the inside of the square root should vanish. We can rewrite the square of the energy as

$$E_{s,-}^2 = v^2(k_x^2 + k_y^2) + (\sqrt{m^2 + v^2k_z^2} - b)^2. \tag{5}$$

For this to vanish, since all terms are positive semi-definite, each must vanish separately. Thus we need $k_x = k_y = 0$ and $k_z^2 = (b^2 - m^2)/v^2$. This can only occur if $b > m$. Then the zeros occur at

$$\vec{K}_\pm = (0, 0, \pm\frac{\sqrt{b^2 - m^2}}{v}). \tag{6}$$

(c) In the case in which there are zero energy states, find the dispersion of the excitations around the "nodal" points, i.e. find the energy spectrum for the low energy states to linear order in $\vec{q} = \vec{k} - \vec{K}_i$. Let $k_x = q_x, k_y = q_y$ and $k_z = K_\pm + q_z$. We have then

$$\begin{aligned} E_{s,-}^2 &= v^2(q_x^2 + q_y^2) + (\sqrt{b^2 + 2v^2K_\pm q_z} - b)^2 \\ &\approx v^2(q_x^2 + q_y^2) + (b + v^2K_\pm q_z/b - b)^2 \\ &= v^2(q_x^2 + q_y^2) + \frac{v^4K_\pm^2}{b^2}q_z^2 \qquad (7) \\ &= v^2(q_x^2 + q_y^2) + v^2\frac{b^2 - m^2}{b^2}q_z^2. \qquad (8) \end{aligned}$$

So we see that the low energy states disperse as

$$E_\pm \approx \pm\sqrt{v^2(q_x^2 + q_y^2) + v_z^2q_z^2}, \tag{9}$$

with $v_z = v\sqrt{b^2 - m^2}/b$. There are two low energy bands which touch at the "Weyl points" and disperse linearly in all directions away from the touching. The Dirac point has split into two Weyl points!