Physics 223b: Problem Set 5

due March 12, 2014 in class, or by email the same day.

1. **Field and energy of vortices:** Let us consider the Ginzburg-Landau free energy, in the limit in which we assume $\psi(\mathbf{x}) = \sqrt{n_s^*}e^{i\theta}$ with *constant* $n_s^* = n_s/2$:

$$F = \int d^3x \left\{ \frac{\hbar^2 n_s}{8m} |\vec{\nabla}\theta - \frac{2e}{\hbar c}\vec{A}|^2 + \frac{|\vec{\nabla} \times \vec{A}|^2}{8\pi} \right\}. \tag{1}$$

   (a) Requiring that the free energy is stationary with respect to variations of $\vec{A}$, derive the partial differential equation for the vector potential $\vec{A}$:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \frac{1}{\lambda^2}\left(\frac{\varphi_0}{2\pi}\vec{\nabla}\theta - \vec{A}\right), \tag{2}$$

   where $\varphi_0 = hc/2e$ is the superconducting flux quantum. You will need to integrate by parts.

   (b) Now use this result to simplify the free energy to

$$F = \int d^3x \left\{ \frac{1}{8\pi}\left(|\vec{B}|^2 + \lambda^2|\vec{\nabla} \times \vec{B}|^2\right) \right\}. \tag{3}$$

   (c) Now taking the curl of Eq. (2) above, show that

$$\vec{B} + \lambda^2 \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{f}(\mathbf{x}), \tag{4}$$

   where naïvely $\vec{f}(\mathbf{x}) = 0$. It can, however, contain delta-function contributions due to the presence of vortices, at the center of which $|\psi| \to 0$ and our initial assumptions broke down.

   (d) Show that for a single vortex line running alone the line $x = y = 0$, if we take $\vec{f}(\mathbf{x}) = f_0\hat{z}\delta(x)\delta(y)$, then $f_0 = hc/2e = \varphi_0$ is required.

   (e) From the above, find the magnetic field distribution around the vortex, i.e. calculate $B(r) = |\vec{B}(\mathbf{x})|$, where $r = \sqrt{x^2+y^2}$ is the distance from the vortex line. Show that

$$B(r) = \frac{\varphi_0}{2\pi\lambda^2}K_0(r/\lambda), \tag{5}$$

   where $K_0(x)$ is the modified Bessel function. Sketch this field distribution. What happens at $r = 0$?

   (f) Now for two vortices with separation d, take $\vec{f}(\mathbf{x}) = \varphi_0\hat{z}\left(\delta(x)\delta(y) + \delta(x-d)\delta(y)\right)$, and show that the free energy takes the form $F = F_0 + U(d)L$, where F_0 is independent of d, and L is the size of the system along the z axis. Show that

$$U(d) = \frac{\varphi_0^2}{8\pi^2\lambda^2}K_0(d/\lambda). \tag{6}$$

2. **d-wave superconductors:** The high-T_c cuprates are d-wave superconductors, with a gap function $\Delta_k \approx \Delta(\cos(k_x a) - \cos(k_y a))$ (a is the lattice spacing of the square lattice). The vanishing of Δ_k for $|k_x| = |k_y|$ leads to various anomalies in their low-temperature behavior. In this problem, model the electronic spectrum as a simple quadratic band $\epsilon_k = k^2/2m$, with $\mu = \epsilon_F = k_F^2/2m$ and $k_F/m \equiv v_F$. Treat the problem as completely two dimensional. You may treat Δ as some experimentally-determined quantity: do not try to work out any self-consistent BCS theory.

   (a) Sketch the quasiparticle density of states $g(\omega)$, for energies $0 < \omega < 4\Delta \ll \epsilon_F$ (You can get a feeling for the shape of the DOS drawing contours of constant energy in $\vec{k}$-space. But I suppose many of you will find a way to do it with Mathematica!). Find an analytical form for its behavior for $\omega \ll \Delta$.

   (b) From this, find the leading term in the low-temperature behavior of the electronic specific heat. Contrast it to the specific heat in a metal, a semiconductor, and an s-wave superconductor.