Physics 223b: Problem Set 5

due March 12, 2014 in class, or by email the same day.

1. **Field and energy of vortices:** Let us consider the Ginzburg-Landau free energy, in the limit in which we assume $\psi(\mathbf{x}) = \sqrt{n_s^*} e^{i\theta}$ with *constant* $n_s^* = n_s/2$:

$$F = \int d^3x \left\{ \frac{\hbar^2 n_s}{8m} |\vec{\nabla}\theta - \frac{2e}{\hbar c}\vec{A}|^2 + \frac{|\vec{\nabla}\times\vec{A}|^2}{8\pi} \right\}. \tag{1}$$

   (a) Requiring that the free energy is stationary with respect to variations of $\vec{A}$, derive the partial differential equation for the vector potential $\vec{A}$:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{1}{\lambda^2}\left(\frac{\varphi_0}{2\pi}\vec{\nabla}\theta - \vec{A}\right), \tag{2}$$

   where $\varphi_0 = hc/2e$ is the superconducting flux quantum.

   By factoring out the $2e/\hbar c$ factor, we can rewrite the free energy as

$$F = \int d^3x \left\{ \frac{1}{8\pi\lambda^2} |\frac{\varphi_0}{2\pi}\vec{\nabla}\theta - \vec{A}|^2 + \frac{|\vec{\nabla}\times\vec{A}|^2}{8\pi} \right\}. \tag{3}$$

   Now let $\vec{A} \to \vec{A} + \delta\vec{A}$, and isolate the terms linear in $\delta\vec{A}$:

$$\delta F = \int d^3x \left\{ -\frac{1}{4\pi\lambda^2}\left(\frac{\varphi_0}{2\pi}\vec{\nabla}\theta - \vec{A}\right)\cdot\vec{\delta A} + \frac{1}{4\pi}\left(\vec{\nabla}\times\vec{\nabla}\times\vec{A}\right)\cdot\delta\vec{A} \right\}. \tag{4}$$

   Requiring that this vanishes for arbitrary $\delta\vec{A}$ gives the desired result.

   (b) Now use this result to simplify the free energy to

$$F = \int d^3x \left\{ \frac{1}{8\pi}\left(|\vec{B}|^2 + \lambda^2|\vec{\nabla}\times\vec{B}|^2\right) \right\}. \tag{5}$$

   Introduce first $\vec{B} = \vec{\nabla}\times\vec{A}$. Then Eq. (2) can be rewritten as

$$\frac{\varphi_0}{2\pi}\vec{\nabla}\theta - \vec{A} = \lambda^2\vec{\nabla}\times\vec{B}. \tag{6}$$

   Substituting this into the first term in Eq. (3) and also using $\vec{\nabla}\times\vec{A} = \vec{B}$ in the second term gives the result directly. Incidentally, no integration by parts is needed. My mistake: that was the previous part!

   (c) Now taking the curl of Eq. (2) above, show that

$$\vec{B} + \lambda^2\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{f}(\mathbf{x}), \tag{7}$$

   where naïvely $\vec{f}(\mathbf{x}) = 0$. It can, however, contain delta-function contributions due to the presence of vortices, at the center of which $|\psi| \to 0$ and our initial assumptions broke down.

Directly taking the curl of Eq. (2), multiplying by λ^2 and using $\vec{\nabla} \times \vec{A} = \vec{B}$, we get

$$\vec{B} + \lambda^2 \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \frac{\varphi_0}{2\pi} \vec{\nabla} \times \vec{\nabla}\theta. \tag{8}$$

The right hand side is $\vec{f}(x)$ which looks like it should vanish, but does not if θ is not single-valued.

(d) Show that for a single vortex line running alone the line $x = y = 0$, if we take $\vec{f}(\mathbf{x}) = f_0 \hat{z}\delta(x)\delta(y)$, then $f_0 = hc/2e = \varphi_0$ is required.

Well we can just see that $\vec{f} = 0$ everywhere except $x = y = 0$ where it is singular, and may contain some delta-function part. To get it, we integral $\int \hat{z} \cdot \vec{f} dx\, dy$ in a neighborhood containing $x = y = 0$. By Stoke's theorem, this gives $\frac{\varphi_0}{2\pi} \oint \vec{\nabla}\theta \cdot d\vec{\ell}$ around the origin which is just equal to φ_0, independent of the contour. From this we deduce the result quoted.

(e) From the above, find the magnetic field distribution around the vortex, i.e. calculate $B(r) = |\vec{B}(\mathbf{x})|$, where $r = \sqrt{x^2 + y^2}$ is the distance from the vortex line. Show that

$$B(r) = \frac{\varphi_0}{2\pi\lambda^2} K_0(r/\lambda), \tag{9}$$

where $K_0(x)$ is the modified Bessel function. Sketch this field distribution. What happens at $r = 0$?

So using $\vec{\nabla} \cdot \vec{B} = 0$ and $B_x = B_y = 0$ and $B_z = B$ by symmetry, Eq. (7) becomes

$$B - \lambda^2 \nabla^2 B = \varphi_0 \delta(x)\delta(y). \tag{10}$$

We can solve this by Fourier transformation. We immediately obtain

$$B(x, y, z) = \int \frac{dk_x dk_y}{(2\pi)^2} \frac{\varphi_0}{1 + \lambda^2(k_x^2 + k_y^2)} e^{i(k_x x + k_y y)}. \tag{11}$$

The solution obviously has $B(x, y, z) = B(r = \sqrt{x^2 + y^2})$ with radial symmetry, and simplifying by choosing $x = r, y = 0$ and letting $k_\mu \to k_\mu/\lambda$, we have

$$B(r) = \frac{\varphi_0}{4\pi^2\lambda^2} \int dk_x dk_y \frac{1}{1 + k_x^2 + k_y^2} e^{ik_x r/\lambda}, \tag{12}$$

and doing the k_y integral gives

$$B(r) = \frac{\varphi_0}{4\pi\lambda^2} \int_{-\infty}^{\infty} dk_x \frac{1}{\sqrt{1 + k_x^2}} e^{ik_x r/\lambda}. \tag{13}$$

The integral is now just twice (because it includes the positive and negative k_x axis) a standard representation for the modified Bessel function. This proves the result. Note that the solution diverges at $r = 0$, but this just means that the assumption $|\psi|$ is constant breaks down.

(f) Now for two vortices with separation d, take $\vec{f}(\mathbf{x}) = \varphi_0 \hat{z} (\delta(x)\delta(y) + \delta(x - d)\delta(y))$, and show that the free energy takes the form $F = F_0 + U(d)L$, where F_0 is independent of d, and L is the size of the system along the z axis. Show that

$$U(d) = \frac{\varphi_0^2}{8\pi^2\lambda^2} K_0(d/\lambda). \tag{14}$$

Since the equation for $\vec{B}$ is linear, we find the solution by simply superimposing the two solutions for the individual delta-function sources. Hence we immediately obtain that

$$B(x,y,z) = B_1 + B_2 \equiv \frac{\varphi_0}{2\pi\lambda^2}\left[K_0(r_1/\lambda) + K_0(r_2/\lambda)\right], \tag{15}$$

where $r_1 = \sqrt{x^2+y^2}$ and $r_2 = \sqrt{(x-d)^2+y^2}$. Now to evaluate the energy it is convenient to write the free energy in Eq. (5) by integration by parts as

$$F = \frac{1}{8\pi}\int d^3x \left\{ \left(\vec{B} + \lambda^2 \vec{\nabla}\times\vec{\nabla}\times\vec{B}\right)\cdot\vec{B} \right\}. \tag{16}$$

Now write $B = B_1 + B_2$ and *keep only the cross terms*, since the terms involving either B_1 or B_2 alone do not depend upon the separation of the vortices. Then we see that

$$F = \text{const} + \frac{1}{8\pi}\int d^3x \left\{ \left(\vec{B}_1 + \lambda^2 \vec{\nabla}\times\vec{\nabla}\times\vec{B}_1\right)\cdot\vec{B}_2 + (1\leftrightarrow 2) \right\}. \tag{17}$$

Now since B_1 and B_2 are independently solutions of Eq. (7), we can replace this by

$$F = \text{const} + \frac{1}{8\pi}\int d^3x \left\{ \varphi_0\delta(x)\delta(y)B_2(x,y) + \varphi_0\delta(x-d)\delta(y)B_1(x,y) \right\}. \tag{18}$$

The two terms give equal contributions, and one obtains the desired result. Note that there was a mistake in the answer given in the homework as stated. The corrected one in this solution is right.

2. **d-wave superconductors:** The high-T_c cuprates are d-wave superconductors, with a gap function $\Delta_k \approx \Delta(\cos(k_x a) - \cos(k_y a))$ (a is the lattice spacing of the square lattice). The vanishing of Δ_k for $|k_x| = |k_y|$ leads to various anomalies in their low-temperature behavior. In this problem, model the electronic spectrum as a simple quadratic band $\epsilon_k = k^2/2m$, with $\mu = \epsilon_F = k_F^2/2m$ and $k_F/m \equiv v_F$. Treat the problem as completely two dimensional. You may treat Δ as some experimentally-determined quantity: do not try to work out any self-consistent BCS theory.

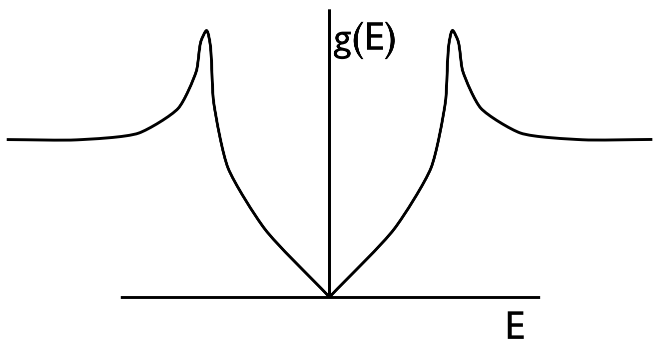
   (a) Sketch the quasiparticle density of states $g(\omega)$, for energies $0 < \omega < 4\Delta \ll \epsilon_F$ (You can get a feeling for the shape of the DOS drawing contours of constant energy in $\vec{k}$-space. But I suppose many of you will find a way to do it with Mathematica!). Find an analytical form for its behavior for $\omega \ll \Delta$.

   So in class we showed that the quasiparticle energies take the form $E_k = \sqrt{(\epsilon_k-\mu)^2 + |\Delta_k|^2}$. This means E_k can be small only when both $\epsilon_k - \mu$ *and* $|\Delta_k|$ are small. This happens only in the vicinity of the nodal points where $k_x = \pm k_y$ crosses the Fermi surface. Let's look near one of these crossings, letting $k_x = 1/sqrt2(k_F + k_1 + k_2)$ and $k_y = 1/sqrt2(k_F + k_1 - k_2)$, with k_1 and k_2 small. If we look at the states near these points, we can approximate $\epsilon_k - \mu \approx v_F k_1)$ and $\Delta_k \approx -\sqrt{2}|\Delta|\sin(k_F a/\sqrt{2})k_2 a \equiv v_\Delta k_2$, with $v_\Delta \equiv -\sqrt{2}|\Delta|\sin(k_F a/\sqrt{2})a$. Hence near this node, we have

$$E_k \approx \sqrt{v_F^2 k_1^2 + v_\Delta^2 k_2^2}. \tag{19}$$

   Thus equal energy contours are ellipses centered around the node. The area inside such an ellipse is given by π times the product of its two radii, i.e. $\pi E^2/(v_F v_\Delta)$. The total number of states with less than energy E is proportional to this area, and so the density of states is proportional to the derivative of this cumulative density of states, hence at low energy, $g(\omega) \sim \omega/(v_F v_\Delta)$ (you do not need to get the prefactor). Thus the low energy density of states vanishes linearly on approaching zero energy.

At high energy, $E \gg |\Delta|$, it must become independent of the gap, since the energy itself is independent of Δ in this limit. There the density of states is constants, as it is for free electrons in two dimensions.



How do these connect? The total integral of the density of states must be unchanged by superconductivity, since this is just the total volume of all k states. So the suppression at small E must be compensated by an enhancement somewhere, which must occur at an energy comparable to Δ. So putting this all together, we would expect something like what is shown in the sketch. Actually the peak diverges, but you need a more detailed analysis to see this.

(b) From this, find the leading term in the low-temperature behavior of the electronic specific heat. Contrast it to the specific heat in a metal, a semiconductor, and an s-wave superconductor.

Since the DOS is linear in energy, we can see that the internal energy $\langle E \rangle \sim \int dE E g(E) f(E) \sim T^3$ ($f(E)$ is the Fermi function), and so $C \sim d\langle E \rangle/dT \sim T^2$. In a metal, we have instead $C \sim \gamma T$ due to the constant density of states, while in a semiconductor and an s-wave superconductor, the electronic specific heat $C \sim e^{-\Delta/k_B T}$ is exponentially suppressed (because in both cases there is a quasiparticle gap).