Most low T_c : $\lambda \simeq 500Å$ $\quad \kappa \simeq \frac{1}{6}$ or less type I
$\xi \simeq 3000Å$

however : Nb, high-T_c $\kappa \gg 1/\sqrt{2}$

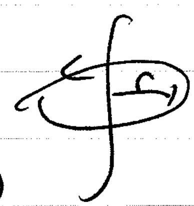
For $\kappa > \frac{1}{\sqrt{2}}$ have "type II" SC.

Then clearly for $H \lesssim H_c$, interfaces will spontaneously form between SC + N regions.

This will happen until the SC is divided into microscopically "small domains".

In fact, instead of domains, Abrikosov showed that what forms is a vortex lattice.

Recall SF vortex



$\oint \vec{v} \cdot d\vec{l}$ is quantized

far from vortex core ($r \gg \xi$)

here $\Psi = \Psi_\infty e^{i\theta(\vec{r})}$ where $\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n$

Free energy $f = \int \frac{\hbar^2 n_s^*}{2m^*} \left(\nabla\theta - \frac{e^*A}{\hbar c}\right)^2 \left(+ \frac{B^2}{8\pi} \right)$

$*$ To minimize $\left(\nabla\theta - \frac{e^*A}{\hbar c}\right)^2$, want $\vec{A} = \frac{\hbar c}{e^*}\vec{\nabla}\theta = \frac{\hbar c}{2e}\nabla\theta$

~~(with be true~~

$\Rightarrow \cancel{\oint A \cdot d\vec{l} =}$

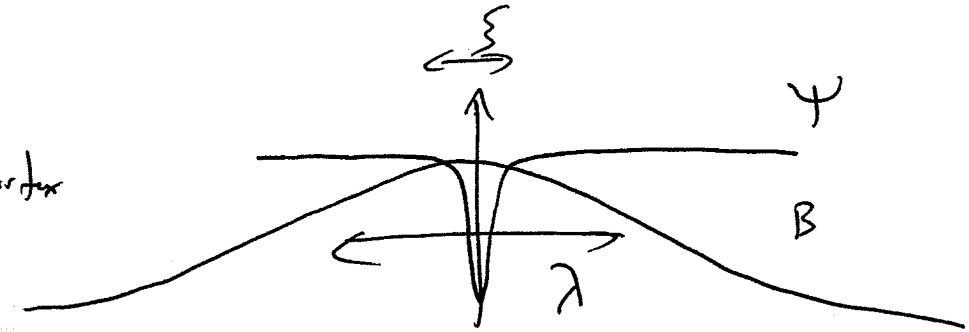
This is always true far enough away from core ($r \gg \lambda$)

Flux Quantum $\Rightarrow \oint \vec{A} \cdot d\vec{l} = \frac{\hbar c}{2e} \int \nabla\theta \cdot d\vec{l} = \frac{hc}{2e} n \equiv n\varphi_0$

$= \int B \cdot d\hat{n}$

note factor of 2! $\quad \varphi_0 = \frac{hc}{2e} \simeq 2 \times 10^{-7}\, G\,cm^2$ SC Flux Quantum.

SC vortex

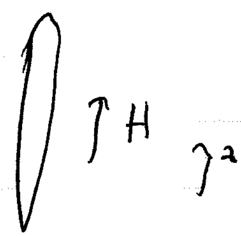


Flux $\int B\, d^2x = \varphi_0 n$

~~energy of a vortex? Most of energy comes from $r > \xi$ where $\psi = \psi_\infty$~~ $\boxed{\kappa \gg 1}$

$$\cancel{g = \frac{\hbar^2 n_s^*}{2m^*}\left(\nabla\theta - \frac{e^*}{\hbar c}\vec{A}\right)^2 + \frac{B^2}{8\pi} - \frac{HB}{4\pi}}$$

~~Most of energy comes from where $r > \xi$~~



Energy of a vortex

$$g = \underbrace{f_{sc} + \frac{B^2}{8\pi}}_{\text{indep. of } H} - \underbrace{\frac{HB}{4\pi}}_{\text{depends on } H}$$

$$\text{So } G = \int d^3r\, g = \underbrace{G_{sc} + G_B}_{\epsilon_v L_z} - \frac{H}{4\pi} L_z \varphi_0$$

$$= \left(\epsilon_v - \frac{H\varphi_0}{4\pi}\right) L_z$$

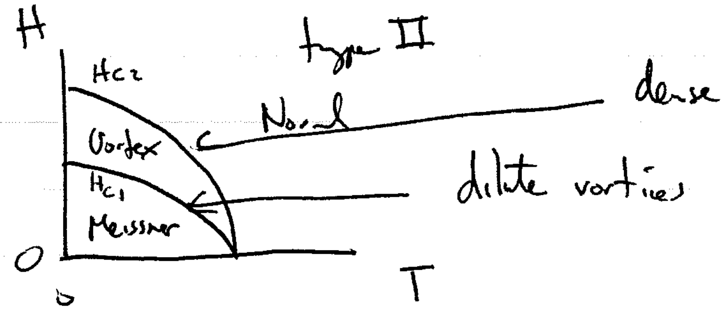
* Vortices will penetrate when $H = H_{c_1} = \dfrac{4\pi \epsilon_v}{\varphi_0}$

ϵ_v generally depends upon full structure of vortex.

You may guess dimensionally $\epsilon_v = \left(H_c^2\, \xi^2\right) f(\kappa)$

For $\kappa \gg 1$ can show $\varepsilon_v = \frac{H_c^2}{8\pi} 4\pi\xi^2 (\ln \kappa) \Rightarrow H_{c1} = \frac{4\pi \varepsilon_v}{\varphi_0}$

otherwise complicated!



Ideal Clean SC
"Abrikosov vortex lattice"
$\downarrow$
otherwise complicated!

near H_{c2}, vortices get close, until typical distance $\sim \xi$.

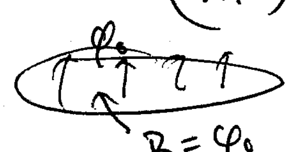
a little rewriting:

$$\frac{H_c^2}{8\pi} = \frac{\alpha^2}{2\beta} \quad (1) \qquad \frac{|\alpha|}{\beta} = \frac{n_s}{2} \quad (3)$$

$$\xi^2 = \frac{\hbar^2}{4m|\alpha|} \quad (2) \qquad \frac{1}{\lambda^2} = \frac{4\pi e^2 n_s}{mc^2} \quad (4)$$

$$\Rightarrow \frac{H_c^2}{8\pi} 4\pi\xi^2 \underset{(1,2)}{=} \frac{\alpha^2}{2\beta} \frac{\pi\hbar^2}{m|\alpha|} = \frac{|\alpha|}{\beta}\frac{\pi\hbar^2}{2m} \underset{(3)}{=} \frac{\pi n_s \hbar^2}{4m} = \left(\frac{\varphi_0}{4\pi\lambda}\right)^2$$

so $\varepsilon_v = \left(\frac{\varphi_0}{4\pi\lambda}\right)^2 \ln\kappa \Rightarrow \boxed{H_{c1} = \frac{\varphi_0}{4\pi\lambda^2}\ln\kappa = \frac{H_c \ln\kappa}{\sqrt{2}\kappa}}$



$\int^{\lambda} d^2r\, B^2 \sim \left(\frac{\varphi_0}{\lambda^2}\right)^2 \lambda^2$ ✓

H_{c2}? Crudely Flux φ_0 per area ξ^2

$H_{c2} \sim \frac{\varphi_0}{\xi^2} = \frac{\varphi_0}{\lambda^2}\kappa^2 \Rightarrow \frac{H_{c2}}{H_{c1}} \propto \kappa^2$

N.B. $\kappa = \frac{1}{\sqrt{2}}$ $H_{c2} = H_{c1} = H_c$ ✓ In fact $\boxed{\frac{H_{c2}}{H_{c1}} = 2\kappa^2}$

I

All this GL theory developed **before** BCS.
(Landau didn't know $e^* = 2e$)

Various experiments which measure
- individual e^- properties
- low-T temperature dependence

require more microscopic understanding.

BCS is a MFT describing the formation of Cooper Pair Condensate out of e^-s (as opposed to GL theory which assumes it).

We will take a simple model of e^-s with some effective attractive interaction.

$$H_{GC} = \sum_k (\varepsilon_k - \mu) C^\dagger_{k\alpha} C_{k\alpha} - \frac{1}{2V} \sum_{k_1+k_2=k_3+k_4} \underbrace{U_{k_1-k_4} C^\dagger_{k_1\alpha} C^\dagger_{k_2\beta} C_{k_3\beta} C_{k_4\alpha}}$$

$= H - \mu N$

attractive

up to constants, this is rewriting of
$$-\sum_q U_q n_q n_{-q}$$
$$= -\int d^3r d^3r' \, U(r-r') n(r) n(r')$$

Idea of MFT $\rightarrow$ want to replace microscopic "pair field" (condensate w.f.) by its average.

Boson = $2e^-$s bound state. $B \sim c_\alpha c_\beta$

- expect Boson to condense in $k=0$ state $B \sim C_{k\alpha} C_{-k\beta}$
- Could form singlet or triplet Bound state.

Singlet $\langle C_{k\alpha} C_{-k\beta} \rangle = \Psi_k \left(\delta_{\alpha\uparrow}\delta_{\beta\downarrow} - \delta_{\alpha\downarrow}\delta_{\beta\uparrow} \right) \leftarrow$ usual BCS

Triplet $\langle C_{k\alpha} C_{-k\beta} \rangle = \Psi_k \left(\begin{matrix} \delta_{\alpha\uparrow}\delta_{\beta\uparrow} \\ \delta_{\alpha\uparrow}\delta_{\beta\downarrow} + \delta_{\alpha\downarrow}\delta_{\beta\uparrow} \\ \delta_{\alpha\downarrow}\delta_{\beta\downarrow} \end{matrix} \right) \leftarrow {}^3He$

<u>Statistics</u> : Singlet $\Psi_k = \Psi_{-k}$
Triplet $\Psi_k = -\Psi_{-k}$

$\Psi_k \sim$ F.T (Relative WF of Cooper Pair)

Could ~~??~~ be orbital ang. momentum $L=0$, $L=1$, $L=2$ state.

Most SCs Singlet, $L=0$ "s-wave"
high-Tc " "$L=2$" ~~Triplet~~ "d-wave"
${}^3He$ Triplet $L=1$

Singlet state: $\langle c_{k\alpha} c_{k'\beta} \rangle = \Psi_k$

$$H_{MF} = \sum_k (\varepsilon_k - \mu) c^\dagger_{k\alpha} c_{k\alpha} - \frac{1}{2V} \sum_{\substack{k_1+k_2\\=k_3+k_4}}' U_{k_1-k_4} \Big[\langle c^\dagger_{k_1\alpha} c^\dagger_{k_2\beta} \rangle c_{k_3\beta} c_{k_4\alpha} + c^\dagger_{k_1\alpha} c^\dagger_{k_2\beta} \langle c_{k_3\beta} c_{k_4\alpha} \rangle - \langle\ \rangle\langle\ \rangle \Big]$$

$$= \sum_k (\varepsilon_k - \mu) c^\dagger_{k\alpha} c_{k\alpha} - \frac{2}{2V} \Big\{ \sum_{k_1,k_4} U_{k_1-k_4} \Big[(\Psi^*_{k_1}) c_{-k_4\uparrow} c_{k_4\downarrow} + \Psi_{k_4} c^\dagger_{k_1\downarrow} c^\dagger_{-k_1\uparrow} - \Psi^*_{k_1} \Psi_{k_4} \Big] \Big\}$$

$$= \sum_k (\varepsilon_k - \mu) c^\dagger_{k\alpha} c_{k\alpha} - \frac{1}{V} \sum_{kk'} \Big[U_{k+k'} \Psi^*_{k'} c_{k\uparrow} c_{-k\downarrow} + U_{k+k'} \Psi_{k'} c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} - U_{k+k'} \Psi^*_k \Psi_{k'} \Big]$$

$$H_{MF} = \sum_k \Big[(\varepsilon_k - \mu) c^\dagger_{k\alpha} c_{k\alpha} - \Delta^*_k c_{k\uparrow} c_{-k\downarrow} - \Delta_k c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} \Big] + \underbrace{\sum_k \Psi^*_k \Delta_k}_{\text{constant}}$$

with $\Delta_k = \frac{1}{V} \sum_{k'} U_{k+k'} \Psi_{k'}$

$$\Leftrightarrow \Delta_k = \frac{1}{V} \sum_{k'} U_{k+k'} \langle c_{k'\uparrow} c_{-k'\downarrow} \rangle$$

$$= \frac{1}{V} \sum_{k'} U_{k-k'} \langle c_{k'\uparrow} c_{-k'\downarrow} \rangle$$

Like ~~Curie~~ Curie-Weiss MFT, this is a self-consistent equation.

To solve it, i.e. calculate $\langle c_{k\uparrow} c_{-k\downarrow} \rangle$, we need to diagonalize H_{MF}

• "Trick": Nambu operators

$$d_{k\uparrow} = c_{k\uparrow}$$
$$d_{k\downarrow} = c^\dagger_{-k\downarrow}$$

$$H = \sum_k \left[(\varepsilon_k - \mu)\left(d^\dagger_{k\uparrow} d_{k\uparrow} + d_{-k\downarrow} d^\dagger_{-k\downarrow}\right) - \Delta_k^* d_{k\uparrow} d^\dagger_{k\downarrow} - \Delta_k d_{k\downarrow} d^\dagger_{k\uparrow} \right]$$

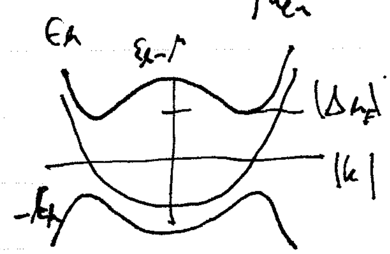
$$= \sum_k d^\dagger_{k\alpha} \underbrace{\left[(\varepsilon_k - \mu)\sigma^z_{\alpha\beta} + \underbrace{\Delta_k \sigma^+_{\alpha\beta} + \Delta_k^* \sigma^-_{\alpha\beta}}_{(\mathrm{Re}\,\Delta_k)\sigma^x_{\alpha\beta} + (\mathrm{Im}\,\Delta_k)\sigma^y_{\alpha\beta}} \right]}_{(\underbrace{\mathrm{Re}\,\Delta_k,\ \mathrm{Im}\,\Delta_k,\ \varepsilon_k - \mu}_{\vec{v}_k}) \cdot \vec{\sigma}} d_{k\beta} + \text{const}$$

Clearly, can choose "quantization axis" along $\vec{v}_k$.

~~So with~~ i.e. $d_{k\alpha} = \phi^+_{k\alpha} d_{k+} + \phi^-_{k\alpha} d_{k-}$

where $\vec{v}_k \cdot \vec{\sigma}_{\alpha\beta} \phi^\pm_{k\beta} = \pm |v_k| \phi^\pm_{k\alpha}$

Then $$H = \sum_k \underbrace{\sqrt{(\varepsilon_k - \mu)^2 + |\Delta_k|^2}}_{E_k} \left(d^\dagger_{k+} d_{k+} - d^\dagger_{k-} d_{k-}\right)$$



Δ_k is called "Gap Function"

Often (in s-wave SC) approx. as k-indep. $\rightarrow$ $\Delta_k = \Delta$ is energy Gap.

Physics: e^-s bound into Cooper pairs, Δ is energy required to unbind one of these.

Ground State: Fill d_{k-} states, d_{k+} states empty.

i.e. $|GS\rangle = \prod_k d^\dagger_{k-} |v\rangle$

What are these excitations like? e.g. Δ_k real.

$$H = \sum_k d^\dagger_k \left[(\epsilon_k - \mu)\sigma^z + \Delta_k \sigma^x \right] d_k = \sum_k d^\dagger_k \begin{pmatrix} \epsilon_k - \mu & \Delta \\ \Delta & -(\epsilon_k - \mu) \end{pmatrix} d_k$$

eigenstates are mixtures of $\uparrow$ + $\downarrow$.

e.g.
$$d_{k+} = u_k d_{k\uparrow} - v_k d_{k\downarrow} = u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow}$$
$$d_{k-} = u_k d_{k\downarrow} + v_k d_{k\uparrow} = u_k c^\dagger_{-k\downarrow} + v_k c_{k\uparrow}$$

So excitations are
$$d^\dagger_{k+} |GS\rangle = (u_k c^\dagger_{k\uparrow} - v_k c_{-k\downarrow}) |GS\rangle$$
and
$$d_{k-} |GS\rangle = (u_k c^\dagger_{-k\downarrow} + v_k c_{k\uparrow}) |GS\rangle$$

here
$$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right) \qquad \xi_k = \epsilon_k - \mu$$
$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

*excited states are superpositions of particles & holes.

* Physics: Cooper pairs condensed $\Rightarrow$ in MFT e^- charge is not a good QN.

i.e. $\Psi_k = \langle c_{k\uparrow} c_{-k\downarrow} \rangle$ doesn't conserve particle #

$\Rightarrow$ G.S. has uncertain e^- number.
(like BEC or SF $\rightarrow$ uncertain Boson number).

(We will return to this — of course in finite chunk of SC ~~part~~ charge is conserved. But MFT is actually still a good approx.)

another way to think about it: In BEC of Cooper pairs, a Cooper pair costs zero energy.

So it can "mix" e^- & hole states.

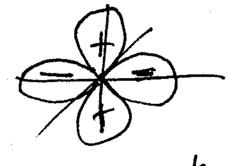
* Note however in singlet SC QPs still are spin eigenstates.

e.g. $d^\dagger_{k\uparrow} = u_k c^\dagger_{k\uparrow} - v_k c_{-k\downarrow}$.

adds $\Delta P = k$, $\Delta S^z = +1/2$; $\Delta P = -(-k) = k$, $\Delta S^z = -(-\tfrac{1}{2}) = +\tfrac{1}{2}$ ✓

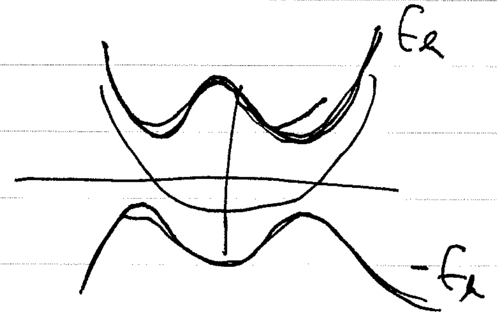
* QP Gap Δ_k is important feature of BCS theory.

* NB. "d-wave" SC $\Delta_{k} \sim \Delta(\cos k_x - \cos k_y)$



$\Rightarrow \Delta_k = 0 \quad |k_x| = |k_y|$
"Nodes" $\Rightarrow$ QP Gap vanishes along these directions

DJS probably first to predict d-wave for cuprates.
Verified by many expts.



OK, Back to self-consistency

$\boxed{T=0}$ G.S. energy

$$E_0 = -\sum_k E_k$$ Fill states w/ $\epsilon < 0$.

Trick:

$$\langle c_{k\uparrow} c_{k\downarrow} \rangle = -\frac{\partial E_0^{MF}}{\partial \Delta_k^*}$$ (P.T.) $\delta E_0 = -\Delta_k^* \langle c_{k\uparrow} c_{k\downarrow} \rangle$

$$= -\frac{\partial}{\partial \Delta_k^*} \sum_k \sqrt{\xi_k^2 + \Delta_k^* \Delta_k} = \frac{\Delta_k}{2\sqrt{\xi_k^2 + |\Delta_k|^2}}$$

SC eqn: $$\Delta_k = \frac{1}{V} \sum_{k'} U_{k-k'} \langle c_{k'\uparrow} c_{k'\downarrow} \rangle = \frac{1}{V} \sum_{k'} U_{k-k'} \frac{\Delta_{k'}}{2\sqrt{\xi_{k'}^2 + |\Delta_{k'}|^2}}$$

Can solve if assume $U_k = U$ const ~~(Reduced approx)~~
and $\Delta_k = \Delta$ const
(Good approx in "weak-coupling" SCs)

$$1 = \frac{U}{2V} \sum_{k'} \frac{1}{\sqrt{\xi_{k'}^2 + |\Delta|^2}} = \frac{U}{2} \int d\varepsilon \frac{N(\varepsilon)}{\sqrt{(\varepsilon-\mu)^2 + |\Delta|^2}}$$

Small $|\Delta| \to$ Controlled by $\varepsilon \simeq \mu = \varepsilon_F$

"Cut-off" $\longrightarrow$ Only states w/in ω_D of ε_F feel attractive interaction

$$1 = \frac{N(\varepsilon_F)U}{2}\int_{-\omega_D}^{\omega_D} d\xi \frac{1}{\sqrt{\xi^2+\Delta^2}} = N(\varepsilon_F)U\int_0^{\omega_D}\frac{d\xi}{\sqrt{\xi^2+\Delta^2}} \simeq N(\varepsilon_F)U \ln\left(\frac{2\omega_D}{\Delta}\right)$$

$$= NU\ln\left(\frac{\omega_D+\sqrt{\omega_D^2+\Delta^2}}{\Delta}\right)$$

$$\Rightarrow \boxed{\Delta \simeq 2\omega_D\, e^{-\frac{1}{N(\varepsilon_F)U}}}$$

BCS Gap at $T=0$.

$T>0$ Behaviour?

$$\langle c_{k\uparrow} c_{-k\downarrow}\rangle = -\frac{\partial F_{MF}(T)}{\partial \Delta_k^*}$$

1 fermion energy ϵ

$F(T)$? $\quad F_{MF}(T) = -k_B T \ln Z \qquad \left(Z_1 = 1+e^{-\beta\epsilon} = \sum_{n=0}^{1} e^{-\beta\epsilon n}\right)$

$$= -k_B T\sum_k\left[\ln\left(1+e^{-\beta E_k}\right) + \ln\left(1+e^{\beta E_k}\right)\right]$$

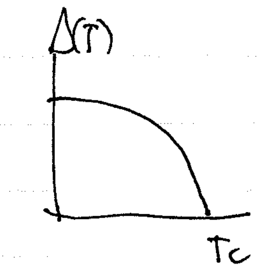
N.B. writing $E_k=\sqrt{\xi^2+|\Delta|^2}$ & expanding in Δ can derive GL Free energy.

$$-\frac{\partial F}{\partial \Delta_k^*} = \sum_k\left(\frac{e^{\beta E_k}}{1+e^{\beta E_k}} - \frac{e^{-\beta E_k}}{1+e^{-\beta E_k}}\right)\frac{\partial E_k}{\partial \Delta_k^*}$$

$$= \sum_k \tanh\left(\beta E_k/2\right)\frac{\partial E_k}{\partial \Delta_k^*} = \sum_k \tanh\left(\beta E_k/2\right)\frac{\Delta_k}{2\sqrt{\xi_k^2+|\Delta_k|^2}}$$

So
$$\Delta_k = \frac{1}{V}\sum_{k'} U_{k-k'}\Delta_{k'}\frac{\tanh\frac{\beta E_{k'}}{2}}{2\sqrt{\xi_{k'}^2+|\Delta_{k'}|^2}\;)E_{k'}}$$

expect



$$\Delta \xrightarrow[T\to T_c]{} 0 \qquad E_k \to \xi_k$$

$$\Delta_k = \frac{1}{V}\sum_{k'} U_{k-k'}\Delta_{k'}\frac{\tanh\frac{\beta|\xi_{k'}|}{2}}{2|\xi_{k'}|}$$

Some approxs:

$$\Delta = \frac{U\Delta}{V}\sum_{k'} \frac{\tanh\frac{\beta|\xi_{k'}|}{2}}{2|\xi_{k'}|} \Rightarrow 1 = \frac{UN(\epsilon_F)}{2}\int_{-\omega_D}^{\omega_D} d\xi \, \frac{\tanh\frac{\beta|\xi|}{2}}{|\xi|}$$

Rescale: $\frac{\beta\xi}{2} = x$

$$1 = UN(\epsilon_F)\int_0^{\beta_c\omega_D/2} dx \left(\frac{\tanh x}{x}\right)$$

This equation determines T_c.

for $\omega_D \gg k_B T_c$ $\quad \int_0^{\beta_c\omega_D/2} dx \frac{\tanh x}{x} \simeq \ln(1.13\,\beta\omega_D)$

$$\Rightarrow \quad k_B T_c \simeq 1.13\,\omega_D\, e^{-1/N(0)V}$$

$\boxed{\Delta(0) = 1.764\, k_B T_c}$ $\longleftarrow$ $\cancel{k_B T_c = 1.764\,\Delta(0)}$

$\rightarrow$ This relation is particular to BCS theory.

Can compare $k_B T_c$ to low temperature ~~spectral~~ QP spectral gap $2\Delta(0)$ which is measured in many ways.

- Tunneling
- Optics ~~(Infrared)~~
- Thermal conductivity
- Specific heat
- T-dep. of penetration depth

less direct $\downarrow$

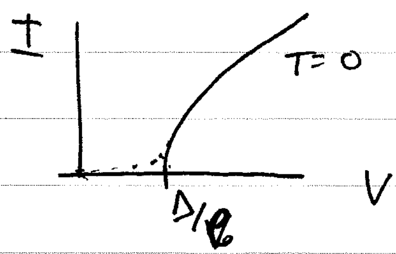
~~Ways to~~

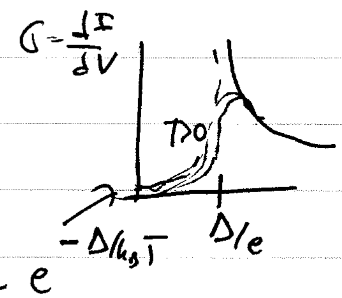
Exptl Consequences of BCS Theory

N-S Tunneling: N $\|$ S, with voltage V across the junction

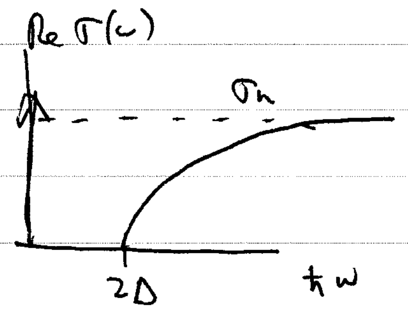
$$H_{tun} = \sum_{kq} t_{kq}\, c^{\dagger}_{kN} c_{qS} + h.c.$$

$I \propto$ Rate $\propto$ 1-pt DOS of e^{-}s with $\varepsilon < V$ in SC.





Optics



Missing weight?

$$\int_0^{\infty} \mathrm{Re}\,\sigma\, d\omega = \frac{\pi n e^2}{2m} \text{ conserved}$$

Goes into $\omega = 0$ peak.

$\approx$ like Drude peak

(Recall $\sigma_{Dru}(\omega) = \dfrac{ne^2/m}{1/\tau - i\omega}$

$\mathrm{Re}\,\sigma = \left(\dfrac{1/\tau\; ne^2/m}{1/\tau^2 + \omega^2}\right)$)

But w/ width $1/\tau \to 0$

(SC e^{-}s don't scatter)

Thermal Conductivity

* Condensate carries no entropy (Just one state).
  Cooper pairs at $p=0$.

* e^- contribution to thermal conductivity (cf. Wiedemann Franz) is due to QPs.

$$\text{Hence} \quad \kappa_e \sim e^{-\Delta/k_B T}$$

Many other properties similarly exponentially suppressed for $T \ll T_c$
- Spec. heat
- $\lambda(0) - \lambda(T)$ (Related to loss of SF density)
  → ~~[illegible]~~
- ⋮

~~Many~~ These can be used to distinguish s-wave from d-wave (ie. detect nodes)

BCS theory also predicts these things for $T \lesssim T_c$ not just $T \ll T_c$.