Like ~~Curie~~ Curie-Weiss MFT, this is a self-consistent equation.

To solve it, i.e. calculate $\langle c_{k\uparrow} c_{-k\downarrow} \rangle$, we need to diagonalize H_{MF}

• "Trick": Nambu operators

$$d_{k\uparrow} = c_{k\uparrow}$$
$$d_{k\downarrow} = c^{\dagger}_{-k\downarrow}$$

$$H = \sum_k \left[(\varepsilon_k - \mu)\left(d^{\dagger}_{k\uparrow} d_{k\uparrow} + d_{-k\downarrow} d^{\dagger}_{-k\downarrow}\right) - \Delta^*_k d_{k\uparrow} d^{\dagger}_{k\downarrow} - \Delta_k d_{k\downarrow} d^{\dagger}_{k\uparrow} \right]$$

$$= \sum_k d^{\dagger}_{k\alpha} \Big[\underbrace{(\varepsilon_k - \mu)\sigma^z_{\alpha\beta} + \underbrace{\Delta_k \sigma^+_{\alpha\beta} + \Delta^*_k \sigma^-_{\alpha\beta}}_{(\mathrm{Re}\,\Delta_k)\sigma^x_{\alpha\beta} + (\mathrm{Im}\,\Delta_k)\sigma^y_{\alpha\beta}}}_{(\mathrm{Re}\,\Delta_k,\ \mathrm{Im}\,\Delta_k,\ \varepsilon_k - \mu)\cdot\vec{\sigma}} \Big] d_{k\beta} + \text{const}$$

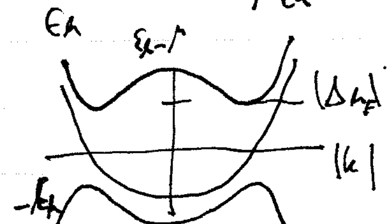
$$\underbrace{(\mathrm{Re}\,\Delta_k,\ \mathrm{Im}\,\Delta_k,\ \varepsilon_k - \mu)}_{\vec{v}_k}\cdot\vec{\sigma}$$

Clearly, can choose "quantization axis" along $\vec{v}_k$.

~~So~~ ~~with~~ i.e. $d_{k\alpha} = \phi^+_{k\alpha} d_{k+} + \phi^-_{k\alpha} d_{k-}$

where $\vec{v}_k \cdot \vec{\sigma}_{\alpha\beta}\, \phi^{\pm}_{k\beta} = \pm |v_k|\, \phi^{\pm}_{k\alpha}$

Then $$H = \sum_k \underbrace{\sqrt{(\varepsilon_k - \mu)^2 + |\Delta_k|^2}}_{E_k} \left(d^{\dagger}_{k+} d_{k+} - d^{\dagger}_{k-} d_{k-}\right)$$



Δ_k is called "Gap Function"

Often (in s-wave SC) approx. as k-indep. $\rightarrow$ $\Delta_k = \Delta$ is energy Gap.

<u>Physics</u>: e^-s bound into Cooper pairs, Δ is energy required to unbind one of these.

~~$d_{k\pm}$~~ Ground State: <u>Fill</u> ~~d_k~~ $k-$ states, $k+$ states empty.

i.e. $|GS\rangle = \prod_k d^\dagger_{k-} |v\rangle$

What are these excitations like? e.g. Δ_k <u>real</u>.

$$H = \sum_k d^\dagger_{k} \left[(\varepsilon_k - \mu)\sigma^z + \Delta_k \sigma^x \right] d_k = \sum_k d^\dagger_k \begin{pmatrix} \varepsilon_k - \mu & \Delta \\ \Delta & -(\varepsilon_k - \mu) \end{pmatrix} d_k$$

eigenstates are mixtures of $\uparrow$ + $\downarrow$.

e.g.
$$d_{k+} = u_k d_{k\uparrow} - v_k d_{k\downarrow} = u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow}$$
$$d_{k-} = u_k d_{k\downarrow} + v_k d_{k\uparrow} = u_k c^\dagger_{-k\downarrow} + v_k c_{k\uparrow}$$

So excitations are
$$d^\dagger_{k+} |GS\rangle = (u_k c^\dagger_{k\uparrow} - v_k c_{-k\downarrow}) |GS\rangle$$
and
$$d_{k-} |GS\rangle = (u_k c^\dagger_{-k\downarrow} + v_k c_{k\uparrow}) |GS\rangle$$

here $u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$ $\qquad \xi_k = \varepsilon_k - \mu$

$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$

*excited states are superpositions of <u>particles</u> & <u>holes</u>.

* <u>Physics</u>: Cooper pairs condensed $\Rightarrow$ in MFT e^- charge is not a good QN.

i.e. $\Psi_k = \langle c_{k\uparrow} c_{-k\downarrow} \rangle$ doesn't conserve particle #

$\Rightarrow$ G.S. has uncertain e^- number.
(like BEC or SF $\to$ uncertain Boson number).

(We will return to this — of course in finite chunk of SC ~~part~~ charge is conserved. But MFT is actually still a good approx.)

another way to think about it: In BEC of Cooper pairs, a Cooper pair costs zero energy.

So it can "mix" e^- & hole states.

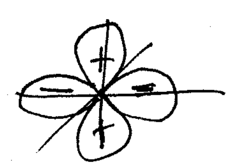
* Note however in singlet SC QPs still are spin eigenstates.

e.g. $d^\dagger_{k\uparrow} = u_k c^\dagger_{k\uparrow} - v_k c_{-k\downarrow}$.

adds $\Delta P = k$, $\Delta S^z = +1/2$ $\qquad$ $\Delta P = -(-k) = k$, $\Delta S^z = -(-\tfrac{1}{2}) = +\tfrac{1}{2}$ ✓

* QP Gap ~~is~~ Δ_k is important feature of BCS theory.

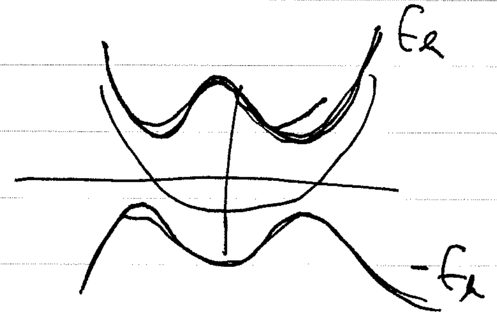
* NB. "d-wave" SC $\Delta_{k} \sim \Delta(\cos k_x - \cos k_y)$



$\Rightarrow \Delta_k = 0$ $\quad |k_x| = |k_y|$

"Nodes" $\Rightarrow$ QP Gap vanishes along these directions

DJS probably first to predict d-wave for cuprates.
Verified by many expts.



OK, Back to self-consistency

$\boxed{T=0}$ G.S. energy.

$$E_0 = -\sum_k E_k \qquad \text{Fill states w/ } \epsilon < 0.$$

Trick:

$$\langle c_{k\uparrow} c_{k\downarrow} \rangle = -\frac{\partial E_0^{MF}}{\partial \Delta_k^*} \qquad (\text{P.T.}) \quad \delta E_0 = -\Delta_k^* \langle c_{k\uparrow} c_{k\downarrow} \rangle$$

$$= -\frac{\partial}{\partial \Delta_k^*} \sum_k \sqrt{\xi_k^2 + \Delta_k^* \Delta_k} = \frac{\Delta_k}{2\sqrt{\xi_k^2 + |\Delta_k|^2}}$$

SC eqn: $\Delta_k = \frac{1}{V} \sum_{k'} U_{k-k'} \langle c_{k'\uparrow} c_{k'\downarrow} \rangle = \frac{1}{V} \sum_{k'} U_{k-k'} \frac{\Delta_{k'}}{2\sqrt{\xi_{k'}^2 + |\Delta_{k'}|^2}}$

Can solve if assume $U_k = U$ const ~~[illegible] approx~~
and $\Delta_k = \Delta$ const
(Good approx in "weak-coupling" SCs)

$$1 = \frac{U}{2V} \sum_{k'} \frac{1}{\sqrt{\xi_{k'}^2 + |\Delta|^2}} = \frac{U}{2} \int d\varepsilon \frac{N(\varepsilon)}{\sqrt{(\varepsilon - \mu)^2 + |\Delta|^2}}$$

Small $|\Delta| \rightarrow$ Controlled by $\varepsilon \simeq \mu = \varepsilon_F$

"Cut-off" $\longrightarrow$ Only states w/in ω_D of ε_F feel attractive interaction

$$1 = \frac{N(\varepsilon_F)U}{2}\int_{-\omega_D}^{\omega_D} d\xi \, \frac{1}{\sqrt{\xi^2+\Delta^2}} = N(\varepsilon_F)\, U \int_0^{\omega_D}\frac{d\xi}{\sqrt{\xi^2+\Delta^2}} \simeq N(\varepsilon_F)\, U \ln\left(\frac{2\omega_D}{\Delta}\right)$$

$$= NU \ln\left(\frac{\omega_D+\sqrt{\omega_D^2+\Delta^2}}{\Delta}\right)$$

$$\Rightarrow \boxed{\Delta \simeq 2\omega_D\, e^{-\frac{1}{N(\varepsilon_F)U}}}$$

BCS Gap at $T=0$.

$T>0$ Behaviour?

$$\langle c_{k\uparrow} c_{-k\downarrow}\rangle = -\frac{\partial F_{MF}(T)}{\partial \Delta_k^*}$$

1 fermion energy ϵ

$$\left(Z_1 = 1+e^{-\beta\epsilon}\right) = \sum_{n=0}^{1} e^{-\beta\epsilon n}$$

$F(T)$?

$$F_{MF}(T) = -k_B T \ln Z$$
$$= -k_B T\sum_k\left[\ln\left(1+e^{-\beta E_k}\right) + \ln\left(1+e^{\beta E_k}\right)\right]$$

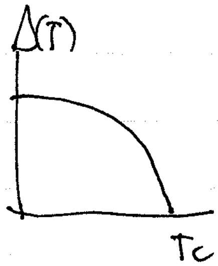
N.B.: writing $E_k=\sqrt{\xi^2+|\Delta|^2}$ & expanding in Δ can derive GL free energy.

$$-\frac{\partial F}{\partial\Delta_k^*} = \cancel{\sum_k}\left(\frac{e^{\beta E_k}}{1+e^{\beta E_k}} - \frac{e^{-\beta E_k}}{1+e^{-\beta E_k}}\right)\frac{\partial E_k}{\partial \Delta_k^*}$$

$$= \cancel{\sum_k}\tanh\left(\beta E_k/2\right)\frac{\partial E_k}{\partial\Delta_k^*} = \cancel{\sum_k}\tanh\left(\beta E_k/2\right)\frac{\Delta_k}{2\sqrt{\xi_k^2+|\Delta_k|^2}}$$

So
$$\Delta_k = \frac{1}{V}\sum_{k'} U_{k-k'}\,\Delta_{k'}\,\frac{\tanh\frac{\beta E_{k'}}{2}}{2\underbrace{\sqrt{\xi_{k'}^2+|\Delta_{k'}|^2}}_{E_{k'}}}$$

expect



$$\Delta \xrightarrow[T\to T_c]{} 0 \qquad E_k \to \xi_k$$

$$\Delta_k = \frac{1}{V}\sum_{k'} U_{k-k'}\,\Delta_{k'}\,\frac{\tanh\frac{\beta|\xi_{k'}|}{2}}{2|\xi_{k'}|}$$

Some approxs:

$$\Delta = \frac{U\Delta}{V}\sum_{k'} \frac{\tanh\frac{\beta|\xi_{k'}|}{2}}{2|\xi_{k'}|} \Rightarrow 1 = \frac{UN(\epsilon_F)}{2}\int_{-\omega_D}^{\omega_D} d\xi \frac{\tanh\frac{\beta|\xi|}{2}}{|\xi|}$$

Rescale: $\frac{\beta\xi}{2} = x$

$$1 = UN(\epsilon_F)\int_0^{\beta_c\omega_D/2} dx \left(\frac{\tanh x}{x}\right)$$

This equation determines T_c.

for $\omega_D \gg k_B T_c$ $\int_0^{\beta_c\omega_D/2} dx \frac{\tanh x}{x} \simeq \ln(1.13\,\beta\omega_D)$

$$\Rightarrow \cancel{\Delta(0)}\ k_B T_c \simeq 1.13\,\omega_D\, e^{-1/N(0)V}$$

$$\boxed{\Delta(0) = 1.764\, k_B T_c} \longleftarrow \cancel{k_B T_c = 1.764\,\Delta(0)}$$

This relation is particular to BCS Theory.
Can compare $k_B T_c$ to low temperature ~~spectral~~ QP spectral gap $2\Delta(0)$ which is measured in many ways.

less direct
- Tunneling
- Optics ~~(Infrared)~~
- Thermal conductivity
- Specific heat
- T-dep. of penetration depth