Hence expect $\quad E_J \sim t^2 \quad$ where t is tunneling amplitude of single e^-.

But note (Recall 223a) tunnel conductance of 2 normal metals across barrier is $\propto$ to 1-e^- tunneling rate

$$1e^- \text{ tunnel rate} \propto |1e^- \text{ amplitude}|^2 \sim t^2$$

So might expect $\quad E_J \propto G_n = R_n^{-1}$ = normal conductance.

Ambegaokar / Baratoff "General" result (many approxs, but good guide)

$$I_c R_n = \frac{\pi \Delta}{2e} \tanh\left(\frac{\Delta}{2k_B T}\right) \qquad \left(\text{ie. } I_c \propto E_J \propto \frac{1}{R_n} \checkmark\right)$$

Drastic consequences!

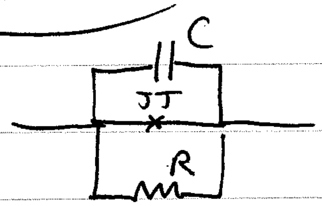
- Zero-bias current
- AC Josephson effect: Put a voltage on $\rightarrow$ AC current

$$\gamma = \frac{2eVt}{\hbar}$$

$$\Rightarrow I = I_c \sin\left(\frac{2eV}{\hbar}\right)t \qquad \omega = \frac{2eV}{\hbar} \; !$$

I-V Curve of a JJ

RCSJ Model



$$I = I_c \sin\gamma + V/R + C\frac{dV}{dt} \qquad V = \frac{\hbar}{2e}\partial_t\gamma$$

$$\cancel{\ddot{\gamma} + Q^{-1}\dot{\gamma} + \sin\gamma = I/I_c}$$

$$\cancel{Q =}$$

((Can also add noise thermal flucts))

$$\frac{C\hbar}{2e}\ddot{\gamma} + \frac{\hbar}{2eR}\dot{\gamma} + I_c\sin\gamma = I$$

$$\ddot{\gamma} + \frac{1}{RC}\dot{\gamma} + \frac{2eI_c}{\hbar C}\sin\gamma = \frac{2eI}{\hbar C}$$

Like pendulum or particle in a $\cos\gamma$ potential + linear in γ "tilted washboard"

~~Simple limit~~

* If $I < I_c$, have solution with $\sin\gamma = I/I_c$ constant. $\dot{\gamma} = \ddot{\gamma} = 0$.

* $I > I_c$. Need dynamics. Simple overdamped limit $\frac{1}{RC}$ large. Compared to characteristic freq.

$$\frac{1}{RC} \gg \sqrt{\frac{2eI_c}{\hbar C}}$$

Can jump to end of next page if necessary.

anyway, then:

$$\frac{1}{R}\dot{\gamma} + \frac{2eI_c}{\hbar}\sin\gamma = \frac{2eI}{\hbar}$$

or $$\dot{\gamma} = \frac{2eI_cR}{\hbar}\left(\frac{I}{I_c} - \sin\gamma\right) > 0$$

$$\int \frac{d\gamma}{I/I_c - \sin\gamma} = \int \frac{2eI_cR}{\hbar}dt = \frac{2eI_cR}{\hbar}t$$

Ask how long it takes to advance γ from 0 to 2π.

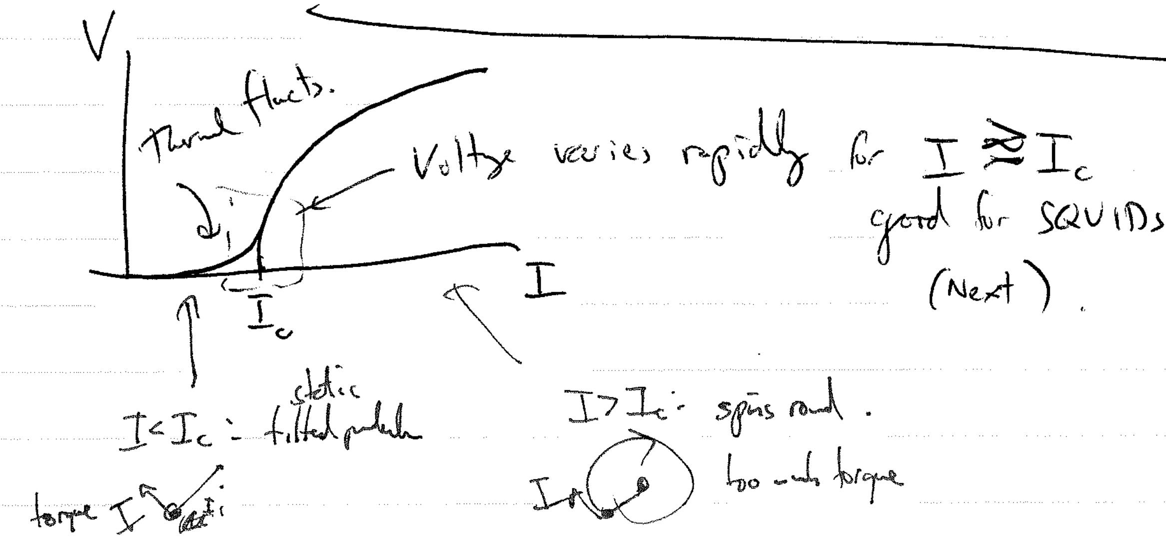
$$\int_0^{2\pi} \frac{d\gamma}{I/I_c - \sin\gamma} = \frac{2\pi}{\sqrt{(I/I_c)^2 - 1}} = \frac{2eI_cR}{\hbar}t_{2\pi}$$

over long time t, $\gamma \simeq 2\pi\left(\frac{t}{t_{2\pi}}\right)$

$$V = \frac{\hbar}{2e}\left\langle\frac{\partial\gamma}{\partial t}\right\rangle_{av} = \frac{\hbar}{2e}\,2\pi/t_{2\pi} = \cancel{\frac{\hbar}{2e}}\,2\pi\left(\frac{\cancel{2e}I_cR}{\cancel{\hbar}}\right)\frac{\sqrt{(I/I_c)^2-1}}{\cancel{2\pi}}$$

(dc voltage)

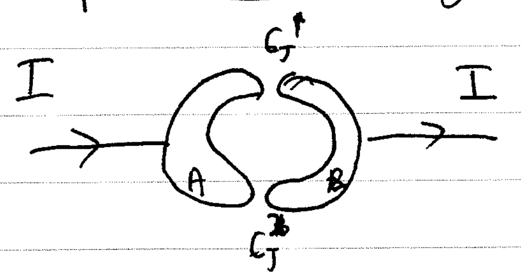
$$\Rightarrow \boxed{V = I_cR\sqrt{(I/I_c)^2 - 1} = R\sqrt{I^2 - I_c^2}}$$



SQUIDs

Many kinds of SQUIDs

* DC - ~~SQ~~ SQUID : Very sensitive magnetometer



Loop ~~enclosing~~ w/ 2 weak links

Can assume supercurrent densities in SCs are negligibly small (since I_c's of JJ's are small by def)

Consider ~~$\oint \vec{J}_s \cdot d\vec{l}$~~ Thus since $\vec{J} \propto \nabla\varphi - \frac{2e}{\hbar c}\vec{A} = 0$

$$\oint \vec{A}\cdot d\vec{l} = \oint_{\substack{\text{inside}\\ SC}} \vec{A}\cdot d\vec{l} + \int_{A_1}^{B_1} \vec{A}\cdot d\vec{l} + \int_{B_2}^{A_2} \vec{A}\cdot d\vec{l}$$

$$= \underbrace{\frac{\hbar c}{2e}\oint_{\substack{\text{inside}\\ SC}} \vec{\nabla}\varphi \cdot d\vec{l}}_{\frac{\hbar c}{2e}\left(\varphi_{A_1} - \varphi_{A_2} + \varphi_{B_2} - \varphi_{B_1}\right)} + \int_{A_1}^{B_1} \vec{A}\cdot d\vec{l} + \int_{B_2}^{A_2} \vec{A}\cdot d\vec{l}$$

$$\Phi = \frac{1}{2\pi}\underbrace{\frac{\hbar c}{2e}}_{\varphi_0}(\gamma_1 - \gamma_2)$$ Gauge-Invariant Phase Diff.

$$\boxed{\Rightarrow \gamma_1 - \gamma_2 = 2\pi\left(\Phi/\varphi_0\right) \quad \text{(defined mod}(2\pi)\text{ of course)}}$$

But phases γ_1, γ_2 are otherwise free.

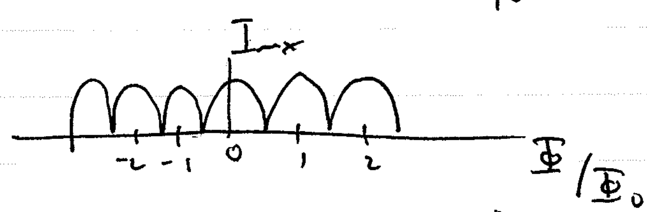
Total SC through ~~#~~ SQUID . Simplest $I_c^1 = I_c^2$.

$$I = I_c \sin\gamma_1 + I_c \sin\gamma_2$$

$$= I_c\left(\sin\left(\frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2}\right) + \sin\left(\frac{\gamma_1+\gamma_2}{2} - \frac{(\gamma_1-\gamma_2)}{2}\right)\right)$$

$$= 2 I_c \cos\frac{\gamma_1-\gamma_2}{2} \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)$$

$$= \underbrace{\left(2 I_c \cos\frac{\pi\Phi}{\varphi_0}\right)}_{I_{max} = 2I_c \left|\cos\frac{\pi\Phi}{\varphi_0}\right| \quad !} \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)$$

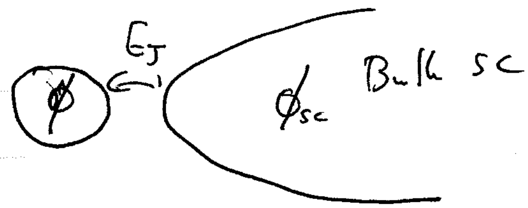


Makes a sensitive magnetometer.

Typically operated with $I > I_{max}$ $(I \simeq 2I_c)$
~~where~~ & measure voltage which then varies drastically w/ flux.

Quantum Effects in JJs + Small SCs

Simplest: Small SC Island (Grain) + Small SC



$\gamma \to \phi$ notation

If grain is isolated, clearly should have *fixed* charge.
On other hand, if E_J large, Cooper-pairs exchanged between SC + Grain.

Should compare "Charging Energy" to E_J

$$H = \frac{1}{2C}(Q - \bar{Q})^2 - E_J \cos(\phi - \phi_{sc}^{0}) \quad \text{arbitrary.}$$

"Background Charge – Set by capacitance to some external gate"

Q ?

Think of grain as a single quantum state w/ N_c bosons (CPs) in it

$$Q = 2eN_c = 2e B^\dagger B$$
$$B = |B| e^{i\phi}$$

~~Think of many bosons on grain (Many CPs).~~ $\cancel{B \sim \sqrt{N_c}\, e^{i\phi}}$

$$Q = CV = \frac{\hbar C}{2e} \partial_t \phi$$

$$H = \frac{\hbar^2 C}{2(2e)^2} \left(\partial_t \phi - \frac{2e}{\hbar C} \overline{Q} \right)^2 - E_J \cos \phi$$

Looks like "particle" with KE $\frac{1}{2} m v^2$

$x \to \phi$
$v \to \partial_t \phi$
$m \to \frac{\hbar^2 C}{(2e)^2}$

conjugate variable $p = mv = \frac{\hbar^2 C}{(2e)^2} \partial_t \phi = \frac{\hbar}{2e} Q$

So ~~[φ,]~~ $[x, p] = i\hbar$

$\downarrow$

$[\phi, \frac{\hbar}{2e} Q] = i\hbar \longrightarrow [\phi, Q] = i\, 2e$

Charge (Number) + Phase are conjugate variables

((Can derive this QMically by considering the grain as one single-particle state occupied by bosons $B, B^\dagger$ with $Q = 2e B^\dagger B$.
$B = |B| e^{i\phi}$))

$\frac{Q^2}{2C}$-term leads to "Quantum-Flucts" of phase.

e.g. ϕ-represent^n $p = \frac{\hbar Q}{2e} = -i\hbar \frac{\partial}{\partial \phi} \Rightarrow Q = -2ie \frac{\partial}{\partial \phi}$

$$H = -\frac{(2e)^2}{2C} \left(\frac{\partial}{\partial \phi} - \frac{i\overline{Q}}{2e} \right)^2 - E_J \cos \phi$$

Can see should compare charging energy $\frac{(2e)^2}{C}$ to E_J.

Suppose $E_J = 0$. $\Psi = e^{iN\phi}$ periodic $\phi \to \phi + 2\pi$

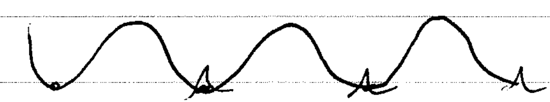
<u>Physics</u>: $Q\Psi = 2eN\Psi$.

$$H = \frac{(2e)^2}{2C}\left(N - \frac{\bar{Q}}{2e}\right)^2$$

discrete levels with definite # of CPs!

$\to$ ϕ is completely uncertain.

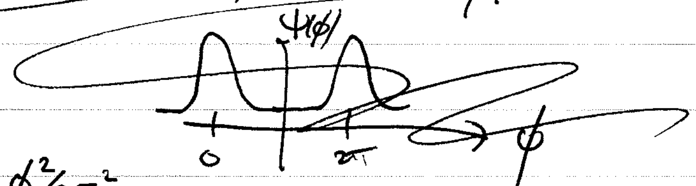
<u>Large E_J?</u> Looks like Hamiltonian of <u>particle</u> with potential $\sim E_J$
$(\bar{Q} = 0)$ + Mass $\sim \frac{C}{(2e)^2}$.



almost have state with minimum ϕ values $(= 2\pi)$.

but with some Quant. Fluct. in each well.

~~but hopefully will have tend to a band and ultimately a state of indefinite ϕ.~~

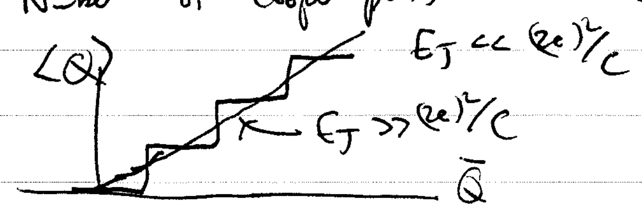


expect $\Psi_0 \sim e^{-\phi^2/2\sigma^2}$

$$\propto \sum_N e^{iN\phi} e^{-\sigma^2 N^2/2}$$ for small $\sigma << 2\pi$!

* very large fluct. in Number of Cooper pairs on Island.

Can study evolution of $\langle Q \rangle$ vs $\bar{Q}$



This can be measured capacitively
or better by tunneling
other means.

N.B. Discrete levels in general can be used to form
"2-level systems" or Qubits.

Various schemes