(1). TIs → ~~sts~~ Ins. not deformable from one to another

ie not deformable to just a product state of localized orbitals

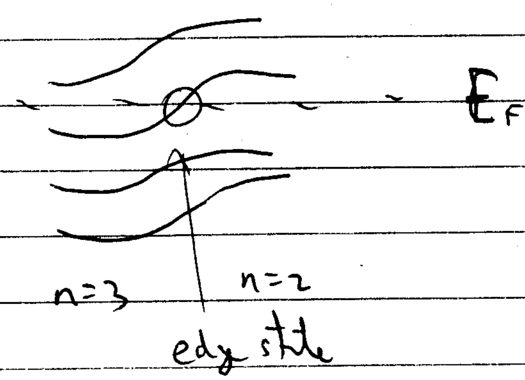
Related to localization of Wannier states. Sometimes they cannot all be localized.

Topology def.

In physics, smooth ≈ adiabatic ≈ maintaining gap.

So we can guess interface between 2 ~~sts~~ topol-distinct insulators must be gapless. "Protected" gapless states

(2) History: IQHE → topological ~~state~~ insulator in B-field (breaks TR)
"Topol. Index" — ~~Invar~~ Invariant which does not change under smooth def. = # of filled LLs
$= \sigma_{xy} / (e^2/h)$



Vacuum ⟺ n = 0.
or ordinary insulator

Kubo formula ~~expr~~ → Chern #.
Explicit expression in terms of Bloch fns.

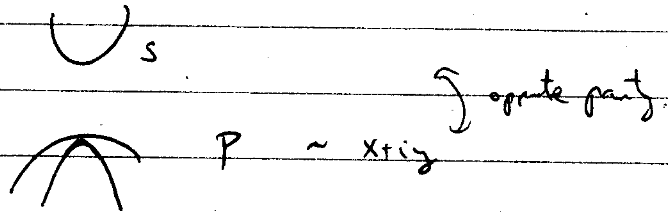
But : Chern # = 0 w/ TRS

So → no topology? Wrong

2005 Kane & Mele (2d)
2006 BHZ → HgTe/CdTe (2d)
2007 3d TIs (theory)
2007 ~~2d~~ HgTe expt.
2008 $Bi_{1-x}Sb_x$ 3d TI expt.

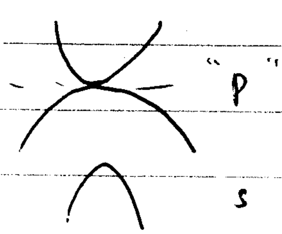
(3) ~~Ele~~ Constructive example – Band Inversion

Let us consider situation in a semiconductor where there is inversion symmetry (not necessary), and thus bands at $k \leftrightarrow -k$ can be labeled by parity. Some bands have $P=+1$ others $P=-1$. Suppose 2 bands with opposite parity are nearby. This is not ~~so~~ uncommon. Usual semiconductors at Γ



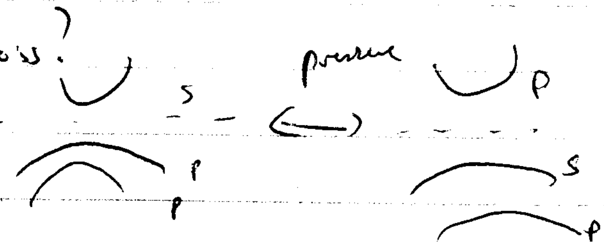
In some materials, these states get close, even exchange order

HgTe



) touching here is a complication. Due to high symmetry of ZincBlende

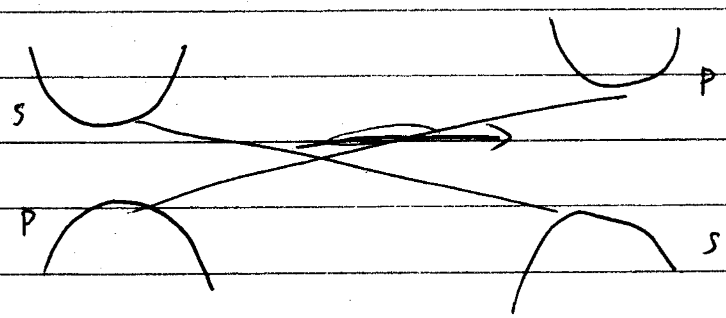
What happens ~~after~~ when these states cross?
Can imagine simpler band structure



* This happens in HgTe/CdTe QWs (2d), Bi_2Se_3 Sb_2Te_3 Bi_2Te_3 (3d)
* Looks promising because $P=+1$ & $P=-1$ states can cross at $\vec{k}=0$ ⟹ this smooth deformation ⟹ gapless state.

Apart from inversion, we will not assume any symmetries

So let's look at this more closely



With TR & Inv., all states are 2-fold degenerate (Kramers)

$$k \xrightarrow[TR]{} -k \xrightarrow[P]{} k$$

So we need to keep 4 states.

$$H = \sum_k c_k^\dagger \underline{\mathcal{H}}(k)\, c_k \quad \text{4×4 matrix}$$

$$= \sum_k c^\dagger_{k a\alpha}\, \mathcal{H}_{\substack{ab\\ \alpha\beta}}\, c_{k b\beta}$$

orbital ↑ "spin"/kramers.

We choose our basis so that at $k=0$ it is diagonalized.

④

Time-reversal $\quad \Psi_{a\alpha} \to \varepsilon_{\alpha\beta} \Psi^*_{a\beta} = \Theta \Psi_{a\alpha}$

Θ = TR operator

$\Theta^2 = -1 \qquad \Theta^{-1} = -\Theta$

Demand $\quad \Theta \mathcal{H}(k) \Theta^{-1} = \mathcal{H}(-k)$

We can write a general matrix $\mathcal{H}$ by using a direct product basis $\vec{\tau}$, $\vec{\sigma}$ acting on orbital, spin.

$$\mathcal{H} = \underline{\underline{A}}(\tau) + \underline{\underline{\vec{B}}}(\tau) \cdot \vec{\sigma} \qquad \Theta \vec{\sigma} \Theta^{-1} = -\vec{\sigma}$$

TR $\Rightarrow \qquad A^*_k = A_{-k} \qquad B^*_k = -B_{-k}$

Take $k=0 \qquad \underline{\underline{A}}^* = \underline{\underline{A}} \qquad \underline{\underline{B}}^* = -\underline{\underline{B}}$

+ Hermiticity $\qquad A^\dagger = A \qquad B^\dagger = B$

$$\Rightarrow A^* = A^T = \begin{pmatrix} a & a'' \\ a'' & a' \end{pmatrix} \qquad \vec{B}^* = -\vec{B}^T = \begin{pmatrix} 0 & i\vec{b} \\ -i\vec{b} & 0 \end{pmatrix}$$

$$\mathcal{H}(k=0) = a_0 + a_1 \tau^z + a_2 \tau^x + \vec{b} \cdot \vec{\sigma} \tau^y$$

Parity $\qquad \Psi \to \tau^z \Psi \qquad \tau^z \mathcal{H}(k) \tau^z = \mathcal{H}(-k)$

$$\Rightarrow \quad a_2 = \vec{b} = 0 \quad \Rightarrow \quad \mathcal{H}(k=0) = a_0 + a_1 \tau^z$$

Now let's look at small non-zero $\vec{k}$.

Expect $\underline{\underline{A}}$ $\underline{\underline{B}}$ are linear functions of $\vec{k}$

$$A(\ell) \simeq A(\ell=0) + A^{\mu} k_{\mu}$$
$$\vec{B}(\ell) = \cancel{\vec{B}(0)} + \vec{B}^{\mu} k_{\mu}$$

Sym constraints: $A_{\mu} = -A_{\mu}^{*} = -A_{\mu}^{T}$ TR + Herm.

$$\Rightarrow A_{\mu} = \tau^{y} \alpha_{\mu}$$

$$\vec{B}_{\mu} = \vec{B}_{\mu}^{*} = \vec{B}_{\mu}^{T}$$

$$B_{\mu} = \beta_0^{\mu} + \beta_1^{\mu} \tau^{z} + \beta_2^{\mu} \tau^{x}$$

P: $\tau^{z} A_{\mu} \tau^{z} = -A_{\mu} \longrightarrow A_{\mu} = \alpha_{\mu} \tau^{y}$ ok

$\tau^{z} \vec{B}_{\mu} \tau^{z} = -B_{\mu}$ $\qquad \vec{B}_{\mu} = \beta_{\mu} \tau^{x}$ only

$$\mathcal{H}(\ell) \simeq a_0 + a_1 \tau^{z} + \vec{\alpha} \cdot \vec{k}\, \tau^{y} + \vec{\beta}_{a} \cdot \vec{k}\, \tau^{x} \sigma^{a}$$

In general, 5 matrices appear here $(+ I)$

Γ_4	τ^{z}
Γ_5	τ^{y}
Γ_1	$\tau^{x} \sigma^{x}$
Γ_2	$\tau^{x} \sigma^{y}$
Γ_3	$\tau^{x} \sigma^{z}$

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2 \delta_{ab} \; !$$

Dirac Γ matrices!

By some combination of rotations in $\vec{k}$ space + Γ space, we can write this as

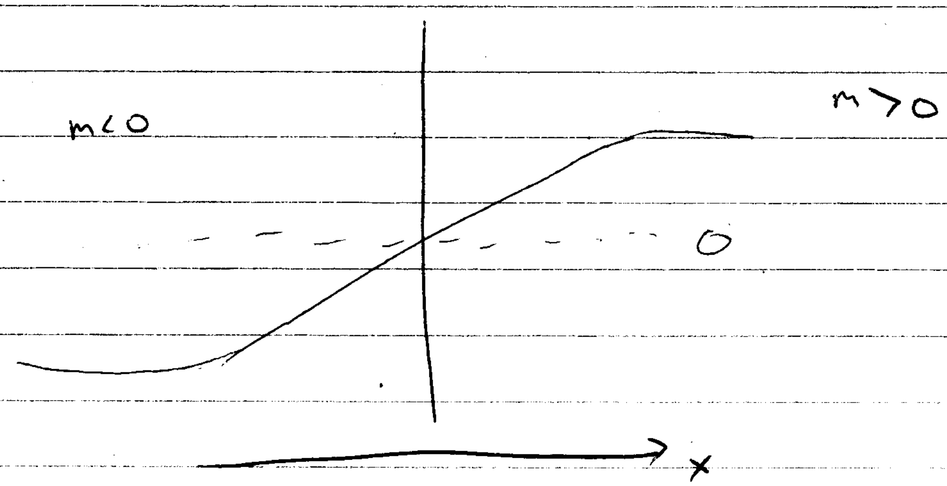
$$\mathcal{H}(\vec{k}) = \underbrace{\cancel{\alpha_0}}_{\text{zero of energy}} + a_1\Gamma_4 + v_1 k_1 \Gamma_1 + v_2 k_2 \Gamma_2 \underbrace{\left(+ v_3 k_3 \Gamma_3 \right)}_{= 0 \text{ in 2d}}$$

$$= \sum_\mu v_\mu k_\mu \Gamma_\mu + m\Gamma_4$$

<u>Dirac</u> Hamiltonian.

Now we can look for the gapless state at the edge of the topol. ins.

Lets look at 2d case.



$$k_x \to -i\partial_x$$

$$\left(-iv_x\Gamma_1\partial_x + v_y k_y \Gamma_2 + m(x)\Gamma_4\right)\Psi = E\Psi$$