$$\left(-iv\tau^y\partial_x + v k_y \tau^x\sigma^x + m(x)\tau^z\right)\psi = \epsilon\psi$$

Let $\sigma^x = \sigma = \pm 1$ eigenstate

$$\left(-iv\tau^y\partial_x + \sigma v k_y\tau^x + m(x)\tau^z\right)\psi = \epsilon\psi$$

Look for $\tau^x = \tau$ eigenstate

$$\tau^x\psi = \tau\psi \qquad \tau = \pm 1$$

$$i\tau^y = \tau^z\tau^x \quad \text{so} \quad i\tau^y\psi = \tau\tau^z\psi$$

So we have

$$\left(\tau^z\left(-v\tau\partial_x + m(x)\right) + \sigma\tau v k_y\right)\psi = \epsilon\psi$$

This is a solution if $\boxed{\epsilon = \sigma\tau v k_y}$

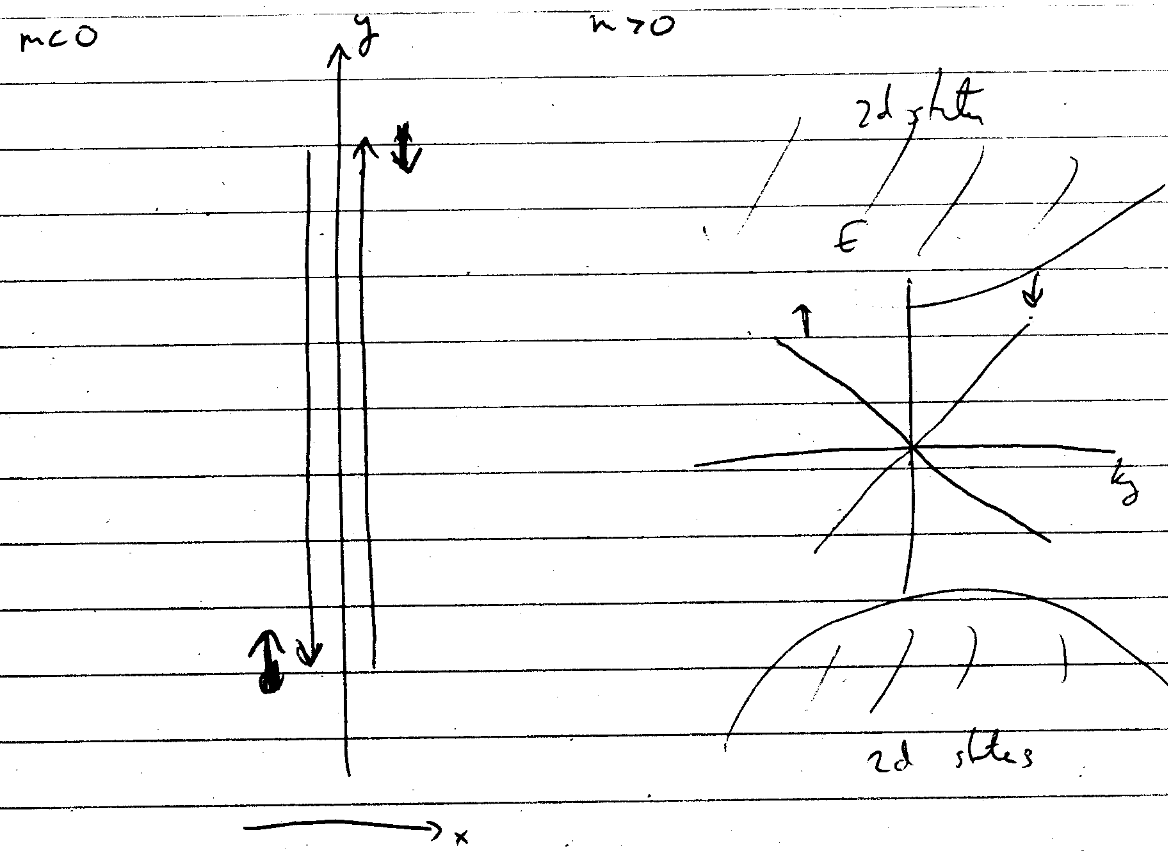
and $\left(-v\tau\partial_x + m(x)\right)\psi = 0$

$$\psi = \psi(0)\, e^{\frac{1}{v\tau}\int_0^x m(x')\,dx'}$$

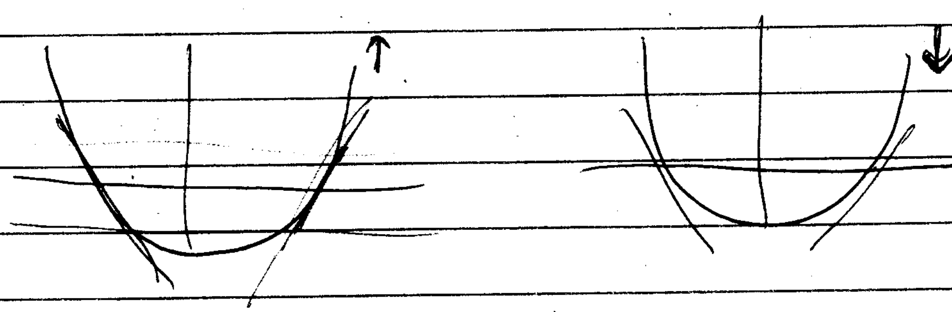
Convergent if $\tau = -1$. Bound state

$$E = -\sigma v k_y$$

8



"Half" of ordinary 1DEG



"Helical" 1DEG

$$H = \sum_k v k_{(y)} \left(\psi_R^\dagger(k)\psi_R(k) - \psi_L^\dagger(k)\psi_L(k) \right)$$

$$\psi_R = \psi_\downarrow \qquad \psi_L = \psi_\uparrow$$

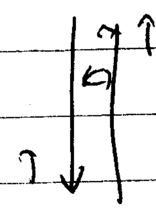
~~Time Reversal~~

$$H = \sum_k -v k\, \psi_k^\dagger \sigma^z \psi_k$$

Time-Reversal: $\vec{\sigma} \rightarrow -\vec{\sigma}$

Usual 1DEG : Backscattering

$$V_{imp}(x)\, \psi_R^\dagger \psi_L$$



Leads to current relaxation, resistivity, localization

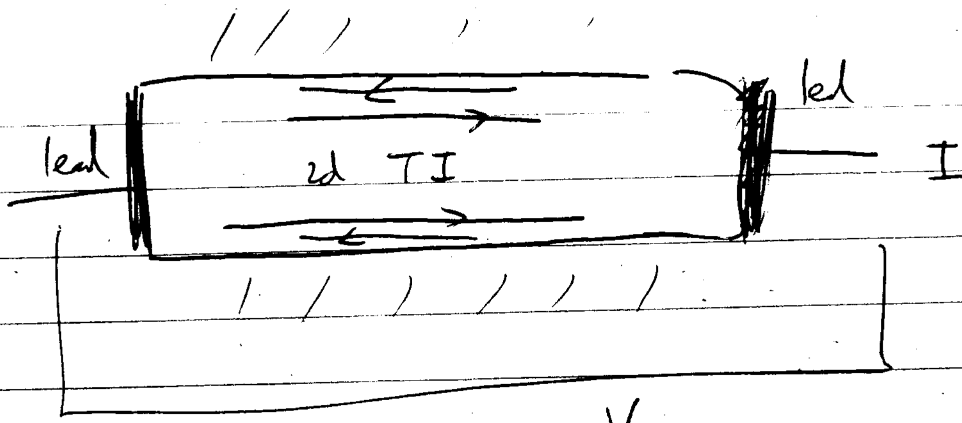
But here such a term is $\cancel{V_{imp}(x)\, \psi^\dagger \sigma^+ \psi}$

Breaks time reversal!
Not possible if TRS preserved.

* Helical 1DEG is very stable to disorder

→ It should, at low temperature, exhibit <u>ideal</u> conductance quantization of a single-channel QWire w/o spin degeneracy

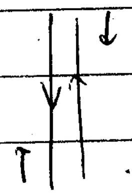
$$G_{\substack{\text{helical}\\ \text{edge}}} = \frac{e^2}{h}$$



$\underline{I} = 2\frac{e^2}{h}V$ parallel channels. [Show Expts?]

Helical edge versus IQHE chiral edge

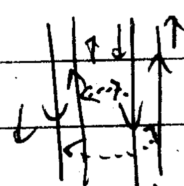
one band inversion



one filled LL

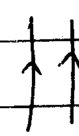
$\nu = 1$

2 band inversion



blue red
non-identical
e's can backscatter
w/o breaking TR
↓
<u>not stable</u>

two filled LLs



still no backscattering
→ stable

$2 \leftrightarrow 0$

$\nu = 2$

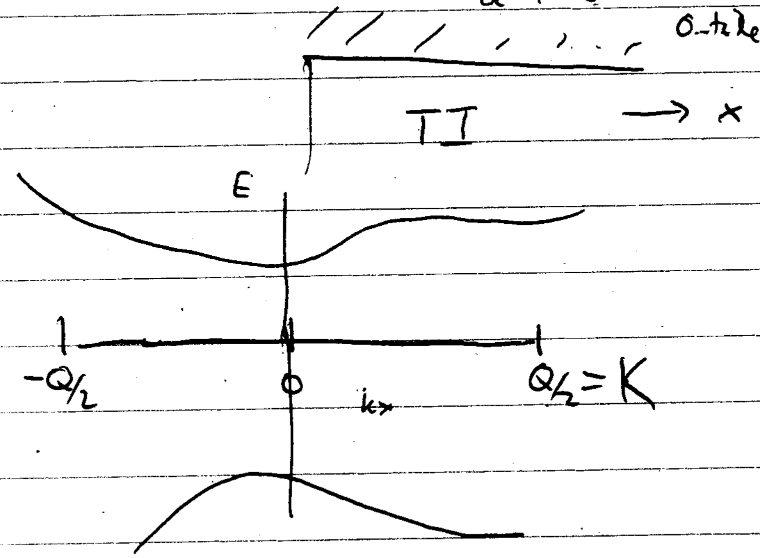
$\mathbb{Z}_2 = \{0, 1\}$ Topological Index
Only parity of edges
is stable.

$\mathbb{Z} = \{\text{Integers}\}$ ~~[illegible]~~ Topological Index

More on $\mathbb{Z}_2$-ness

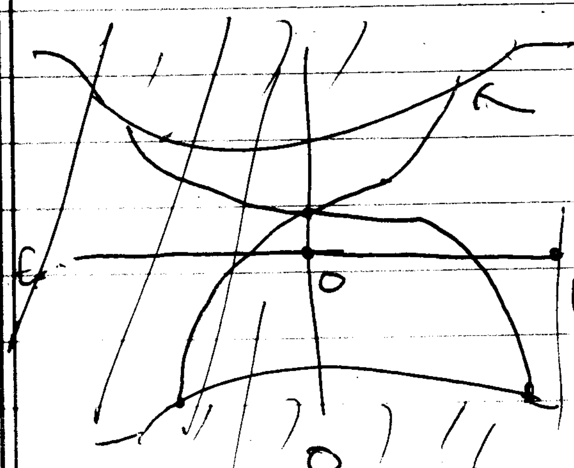
We can consider band inversion ~~at~~ at other TRIM
(TR invariant momenta)

Consider surface BZ — For 2d system, this is
a line



TR implies
at TRIM
($k_x = 0$, $k = Q/2$)
Must have Kramers degeneracy.

Suppose band inversion at $k = 0$ only



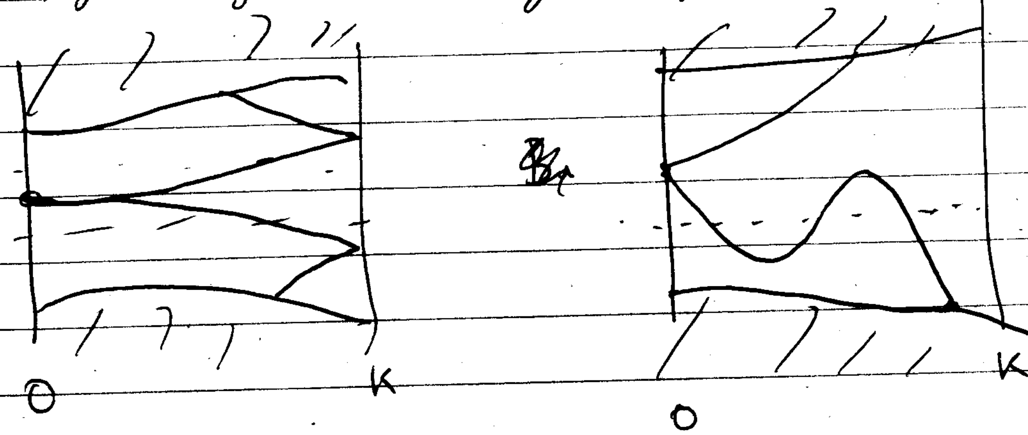
edge states go into continuum

~~Fermi energy~~
One state crosses Fermi energy
going from $k_x = 0 \to K$

Can ignore
since reflection

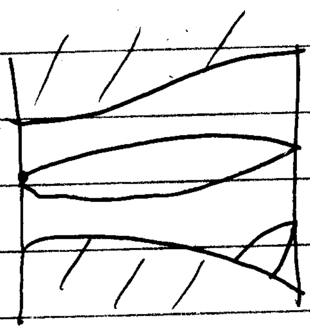
10 B

By smoothly deforming this, can get edge states also at K



__But__: # of crossings of Fermi energy is always __odd__.

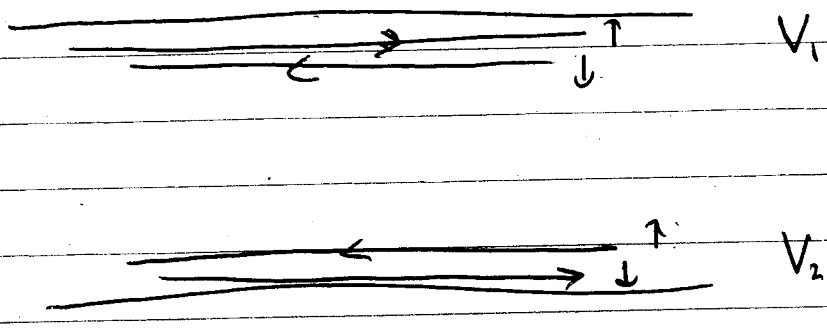
In trivial insulator, # is zero or even

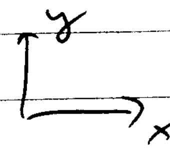


This is another way to see $\mathbb{Z}_2$-ness from surface spectra

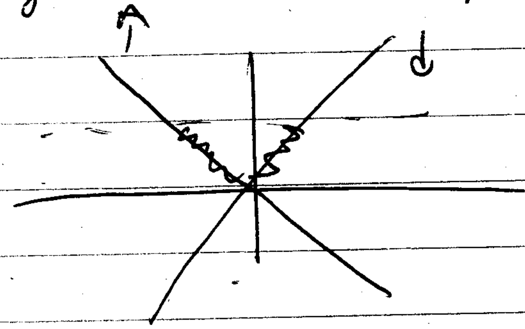
~~QL8~~ Quantum Spin Hall Effect (QSHE)

Original name for 2d TI





each edge if we increase potential $\rightarrow$ no net charge current



But $I^x_z = I^x_\uparrow - I^x_\downarrow \neq 0$
$\propto V$

$$I^x_{top} = \frac{e^2}{h}\frac{\hbar}{2e} V_1$$

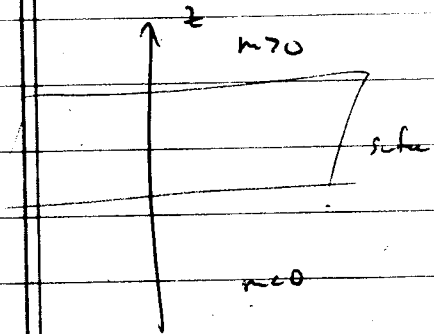
$$I^x_{bot} = -\frac{e^2}{h}\frac{\hbar}{2e} V_2$$

$$\Rightarrow \boxed{I^x_z \sim \frac{e}{4\pi h}(V_1 - V_2) \sim \frac{e}{4\pi h} V_y}$$

"Quantized Spin Hall Effect"

But... hard to detect, since spin itself is not really conserved.

3d Case



$$\mathcal{H}(k) = v k_x \Gamma_1 + v k_y \Gamma_2 - i v \partial_z \Gamma_3 + m(z)\Gamma_4$$

~~Suppose~~ Consider $\Gamma_{34} = -i\Gamma_3\Gamma_4$

$$[\Gamma_{34}, \Gamma_1] = [\Gamma_{34}, \Gamma_2] = 0$$

$$\Gamma_{34} = \Gamma_{34}^\dagger \qquad \Gamma_{34}^2 = 1 \rightarrow \Gamma_{34} = \pm 1 \text{ eigenvalues}$$

Suppose $\Gamma_{34}|\psi\rangle = z|\psi\rangle \qquad z = \pm 1$

$$\cancel{\Gamma_4 = \Gamma_{34}^2 |\psi\rangle = z\,\Gamma_{34}}$$

$$\Gamma_3|\psi\rangle = \Gamma_3\Gamma_{34}^2|\psi\rangle = \Gamma_3\Gamma_{34}\, z|\psi\rangle$$
$$= \Gamma_3(-i)\Gamma_3\Gamma_4\, z|\psi\rangle$$
$$= -iz\Gamma_4|\psi\rangle$$

So
$$\mathcal{H}(k)|\psi\rangle = (v k_x\Gamma_1 + v k_y\Gamma_2)|\psi\rangle + \Gamma_4(-v z\partial_z + m(z))|\psi\rangle$$

So choose

$$(-v\tau\partial_z + m(z))|\psi\rangle = 0$$

$$|\psi\rangle = e^{v\tau\int_0^z m(z)} |u\rangle$$

ok if $\tau < 0$

$|u\rangle$ then satisfies

$$(v k_x \Gamma_1 + v k_y \Gamma_2)|u\rangle = E|u\rangle$$

2d Massless Dirac Equation!

Note: we can take this to be a 2×2 matrix eqn.

Why? Original equation was 4×4. We can choose basis states as eigenstates of

$$(\Gamma_{12}, \Gamma_{34}) = 0 \qquad \Gamma_{12} = \sigma^z = \pm 1$$
$$\Gamma_{34} = \tau^z = \pm 1$$

$|1,1\rangle \quad |-1,1\rangle \quad |-1,-1\rangle |1,-1\rangle$

$\tau = -1 \Rightarrow$ these two states.

$\Gamma_{12} = -i\Gamma_1\Gamma_2$ Same relation as $\sigma^z = -i\sigma^x\sigma^y$

$\Gamma_1 \to \sigma^x \quad \Gamma_2 \to \sigma^y$

$H_{2d} = v k_x \sigma^x + v k_y \sigma^y$!