as $\tau<0$, the two states are given by

$$|1,-1\rangle \quad \& \quad |-1,-1\rangle$$

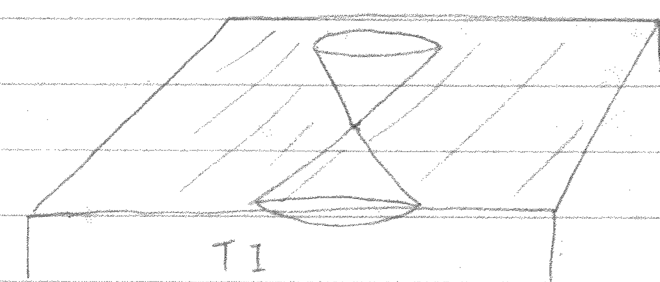
as $\Gamma_{12} = \sigma^z = -i\Gamma_1\Gamma_2 \longleftrightarrow \sigma^z = -i\sigma^x\sigma^y$,

$\Gamma_1 \to \sigma_x$, $\Gamma_2 \to \sigma_y$ gives

$$\underline{H_{2d} = v k_x \sigma_x + v k_y \sigma_y}$$

II Surface State of 3D TI

1.



① Chiral 2D Dirac Fermions.

② $H_{2d} : v k_x \sigma^x + v k_y \sigma^y$
$= v\, k^{(2)} \cdot \sigma^{(2)}$

③ Physical spin S may not be $S \parallel \sigma$. e.g.
$S_x \sim \sigma^y$ & $S_y \sim -\sigma^x$

④ Mass term

Gen. form $H = v k_x \sigma_x + v k_y \sigma_y + m\sigma_z$

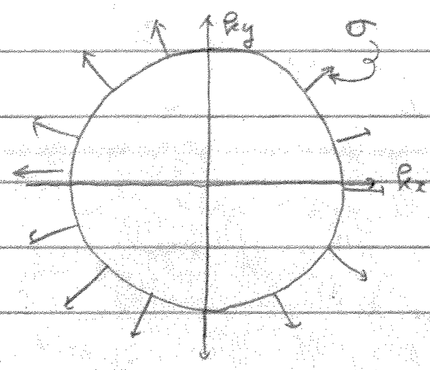
However, $K\sigma^z K^\dagger \to -\sigma^z$ hence $m \neq 0$ breaks TRS

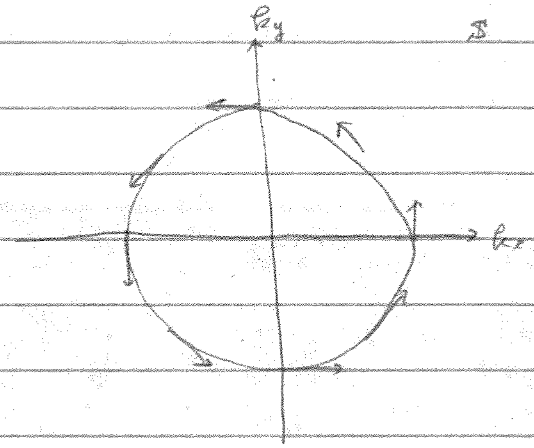
$\to$ Massless for protected TRS.

2. Spin - momentum locking. (Take $E_f > 0$ = Dirac point)

① $\varepsilon = \sqrt{v^2(k_x^2 + k_y^2)}$

cf.) $\mathbf{h} \cdot \boldsymbol{\sigma} = |\mathbf{h}|$





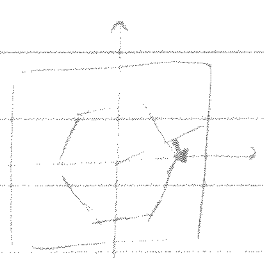
→ uniqeue spin state for each $\mathbf{k}$.

(no spin deg.)

② Spin - res. ARPES

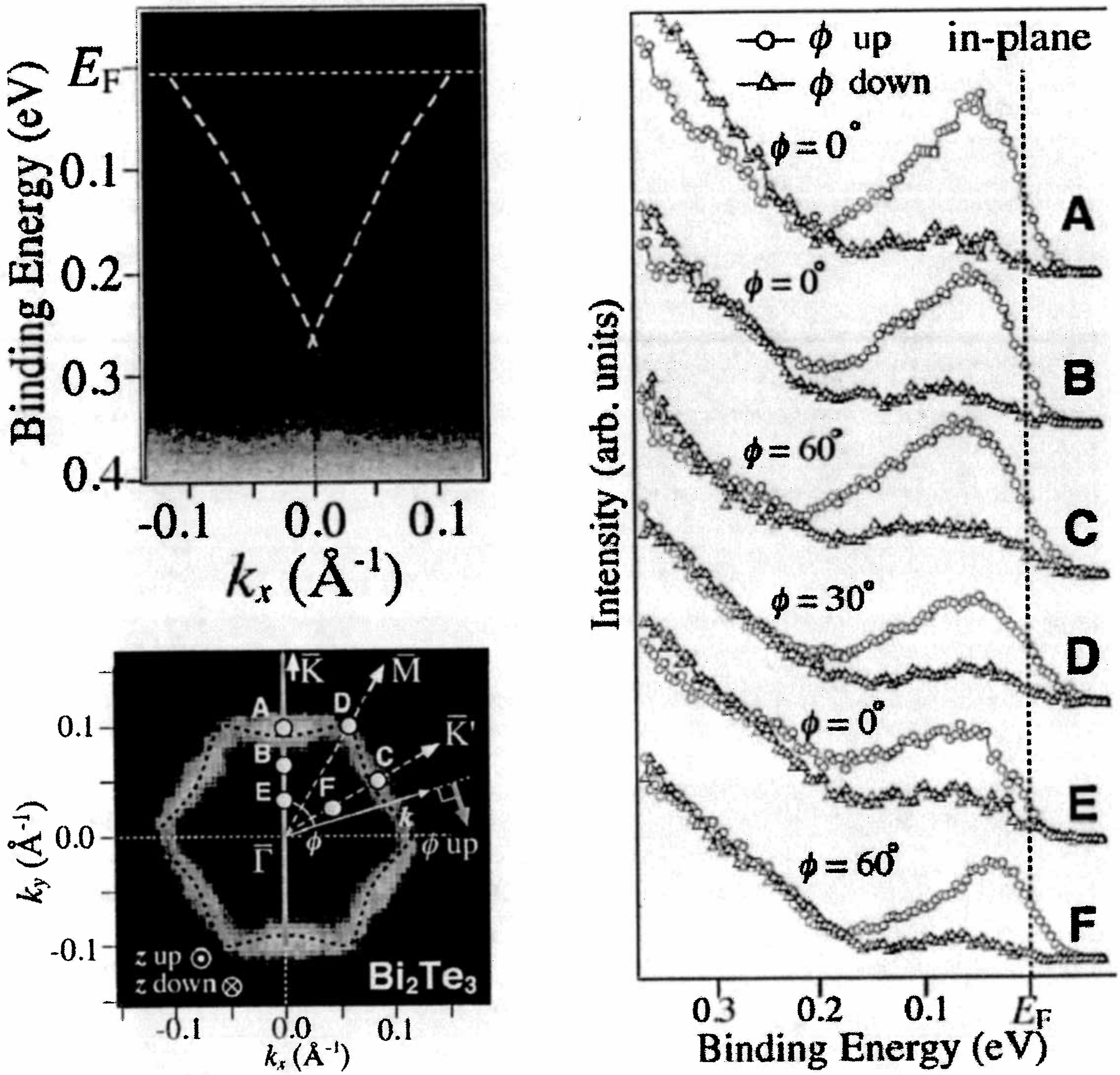
1. Exp. by Japanese Grp. (Tohoku + Osaka) on Bi_2Te_3

2. Left Top : Ene. disp. of Edge mode below Fermi level; You can see the Dirac cone



3. Left Bot Fermi surf. ; not in a perfect circle due to 6-fold symmetry of the lattice

4. Spin res. ARPES at A ~ E in Left - Bot. Figure. Shoulder Below 0.2 eV → Bulk, 0.2 eV ~ 0.0 eV → Edge



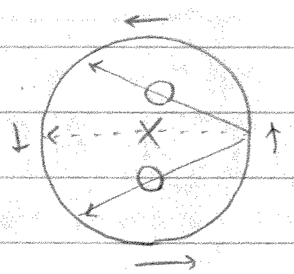
Extracted from S. Souma *et al.*, Phys. Rev. Lett. 106, 216803 (2011)

③ Suppressed back scattering

1. $k \to -k$ $\to$ $\sigma \to -\sigma$

$\to$ breaks TRS.

2. if $k \nrightarrow -k$ $\to$ Scattering is ok w/ TRS.



Ⅲ. Half-integer QHE

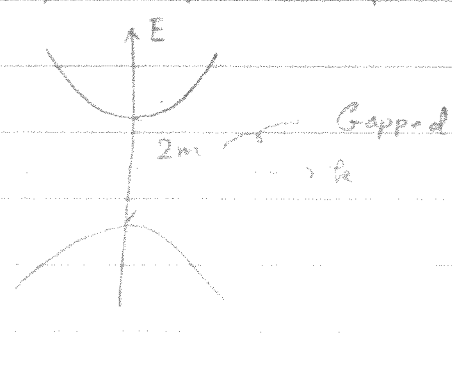
① With TRS, $\sigma_{xy} = 0$.

② w/ weak Zeeman Field // z,

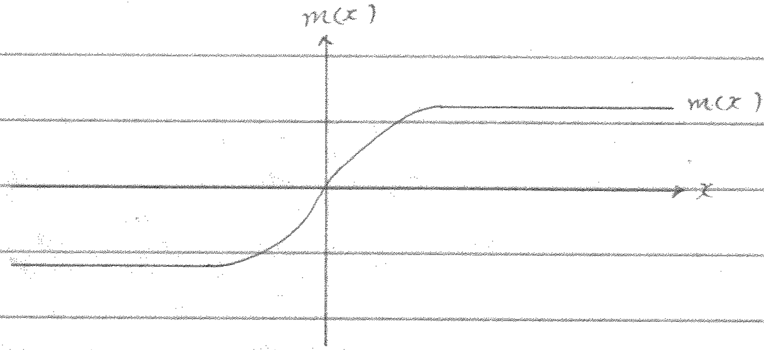
$$H = v k_x \sigma_x + v k_y \sigma_y + m \sigma_z \quad (m \sim g B_z)$$

$$\downarrow$$

$$E(k) = \pm\sqrt{v^2 k^2 + m^2}$$



③ Domain Wall.



1. The same problem we studied for 2D TI, but only half the degrees of freedom

(*) $$\mathcal{H} = v k_y \sigma_y - i v \partial_x \sigma_x + m(x)\sigma_z .$$

$$\downarrow \ \sigma_x = i\sigma_z\sigma_y$$

$$= v k_y \sigma_y + \sigma_z (v \partial_x \sigma_y + m(x))$$

suppose $\sigma_y \psi = \sigma \psi , \quad \sigma = \pm 1$

$$\varepsilon\psi = \mathcal{H}\psi = v\sigma k_y \psi + \sigma_z (\sigma v \partial_x + m(x)) \psi$$

for $\varepsilon = v\sigma k_y$

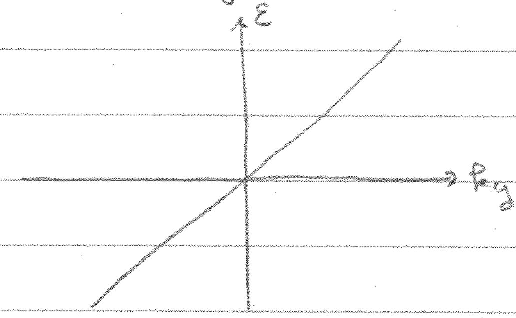
$$(\sigma v \partial_x + m(x)) \psi = 0.$$

$$\rightarrow \quad \psi(x) = \psi(0) \exp\left(-\frac{1}{\sigma v} \int_0^x m(x')dx' \right)$$

if $\sigma = 1$, bound state w/

$$\varepsilon = v\sigma k_y = +v k_y$$

⇒ Chiral Edge Mode



⇒ $\sigma_{xy} = e^2/h$

⇒ as this edge mode is located at the boundary of two TI w/ applied $\mathbb{B}$ in the opposite dirc.,

$$\sigma_{xy} = \sigma_{xy}(m>0) - \sigma_{xy}(m<0) = e^2/h$$

⇒ as $\sigma_{xy}(m)$ is odd func. of m,

$$\sigma_{xy} = \frac{e^2}{2h}\,\mathrm{sgn}(m) \quad : \text{half-Int. QHE}$$

④ Kubo Formula

1. $$\sigma_H = \frac{1}{i\hbar L^2} \sum_{\sigma\neq\sigma'} \sum_{k} \frac{f(\varepsilon_{\sigma k}) - f(\varepsilon_{\sigma' k})}{(\varepsilon_{\sigma k} - \varepsilon_{\sigma' k})^2} \langle \sigma k | \hat{J}_x(k) | \sigma' k \rangle \langle \sigma' k | \hat{J}_y(k) | \sigma k \rangle$$

$$= \frac{1}{i\hbar L^2} \sum_{\sigma=\pm 1} \sum_{k} \frac{f(\varepsilon_{\sigma k}) - f(\varepsilon_{\bar\sigma k})}{(\varepsilon_{\sigma k} - \varepsilon_{\bar\sigma k})^2} \langle \sigma k | \hat{J}_x(k) | \bar\sigma k \rangle \langle \bar\sigma k | \hat{J}_y(k) | \sigma k \rangle$$

$$= \frac{1}{i\hbar L^2} \sum_{\sigma=\pm 1} \sum_{k} f(\varepsilon_{\sigma k}) \frac{\langle \sigma k | \hat{J}_x(k) | \bar\sigma k \rangle \langle \bar\sigma k | \hat{J}_y(k) | \sigma k \rangle - (x \leftrightarrow y)}{(\varepsilon_{\sigma k} - \varepsilon_{\bar\sigma k})^2}$$

here, $J(k) = \partial_k H(k)$, and

$$\langle \sigma k | [\partial_k H(k)] | \sigma' k \rangle = (\mathcal{E}_{\sigma k} - \mathcal{E}_{\sigma' k}) \langle \sigma k | \partial | \sigma' k \rangle$$

$$\circledast \quad \partial_k [H(k) | \sigma' k \rangle] = [\partial_k \mathcal{E}_{\sigma' k}] | \sigma' k \rangle + \mathcal{E}_{\sigma' k} \, \partial_k | \sigma' k \rangle$$

$$= [\partial_k H_k] | \sigma' k \rangle + H(k) \, \partial_k | \sigma' k \rangle$$

$$\underline{\langle \sigma k | [\partial_k \mathcal{E}_{\sigma' k}] | \sigma' k \rangle} + \langle \sigma k | \mathcal{E}_{\sigma' k} \partial_k | \sigma' k \rangle$$

$$= \langle \sigma k | (\partial_k H_k) | \sigma' k \rangle + \mathcal{E}_{\sigma k} \langle \sigma k | \partial_k | \sigma' k \rangle$$

if $\sigma \neq \sigma'$, this vanishes as $\partial_k \mathcal{E}_{\sigma' k}$ a real vector.

hence,

$$\langle \sigma k | (\partial_k H(k)) | \sigma' k \rangle = (\mathcal{E}_{\sigma' k} - \mathcal{E}_{\sigma k}) \langle \sigma k | \partial_k | \sigma' k \rangle$$

$$\sigma_H = \frac{1}{\hbar L^2} \sum_{\substack{\sigma = \pm 1 \\ k}} f(\mathcal{E}_{\sigma k}) \left\{ \langle \sigma k | \partial_{k_x} | \bar{\sigma} k \rangle \langle \bar{\sigma} k | \partial_{k_y} | \sigma k \rangle - (x \leftrightarrow y) \right\}$$

as $\langle \sigma k | \cdot | \sigma k \rangle \langle \sigma k | \cdot | \sigma k \rangle$ vanish for the first & 2nd term.

$$= \frac{1}{\hbar L^2} \sum_{\substack{\sigma, \sigma' \\ k}} f(\mathcal{E}_{\sigma k}) \left\{ \langle \sigma k | \partial_{k_x} | \sigma' k \rangle \langle \sigma' k | \partial_y | \sigma k \rangle - (x \leftrightarrow y) \right\}$$

① $\underbrace{\partial_{k_x} (\langle \sigma k | \sigma k \rangle)}_{0} = (\partial_{k_x} \langle \sigma k |) | \sigma' k \rangle + \langle \sigma k | \partial_{k_x} | \sigma' k \rangle$

② $\langle \sigma k | \partial_{k_x} \partial_{k_y} | \sigma k \rangle = \langle \sigma k | \partial_{k_x} \partial_{k_y} | \sigma k \rangle$

$$= \frac{-1}{\hbar L^2} \sum_{\sigma k} f(\mathcal{E}_{\sigma k}) \left\{ \partial_{k_x} (\langle \sigma k | \partial_{k_y} | \sigma k \rangle) - \langle x \leftrightarrow y \rangle \right\}$$

def $A_\sigma(k) = -i \langle \sigma k | \partial_k | \sigma k \rangle$

then

$$\sigma_H = \nu \frac{e^2}{h} \quad , \quad \nu = \int_{BZ} \frac{d\mathbf{k}}{2\pi} \left[\partial_{k_x} a_{-y}(\mathbf{k}) - \partial_{k_y} a_{-x}(\mathbf{k}) \right]$$

2. For $H = \mathbf{R}(\mathbf{k}) \cdot \sigma$, $\rightarrow \varepsilon = \pm R$.

$$\nabla \times A_\pm(\mathbf{R}) = \pm \frac{1}{2} \frac{\mathbf{R}}{R^3} \quad , \quad A_\pm(\mathbf{R}) = -i \langle \pm \mathbf{R} | \partial_{\mathbf{R}} | \pm \mathbf{R} \rangle$$

in the current case,

$$\mathbf{R} = \begin{pmatrix} v k_x \\ v k_y \\ m \end{pmatrix}$$

hence,

$$-i \langle -1\mathbf{k} | \partial_{k_x} | -1\mathbf{k} \rangle = v(-i) \langle -1\mathbf{k} | \partial_{R_x} | -1\mathbf{k} \rangle = v A_{-x}(\mathbf{R})$$

$$\nu = \int_{BZ} \frac{d\mathbf{k}}{2\pi} \left[v \left(\partial_{R_x} a_{-y}(\mathbf{k}) - \partial_{R_y} a_{-x}(\mathbf{k}) \right) \right]$$

$$= -\frac{1}{4\pi} \int_{BZ} d\mathbf{k} \frac{v^2 m}{(v^2 k^2 + m^2)^{3/2}} = -\frac{1}{2} \int_0^{k_c} dk \frac{v^2 m k}{(v^2 k^2 + m^2)^{3/2}}$$

$$= \frac{1}{2} \int_0^{k_c} \partial_k \cdot \hat{R}_z \quad , \quad \hat{R}_z = \frac{R_z}{R} \quad , \quad R = \sqrt{v^2 k^2 + m^2}$$

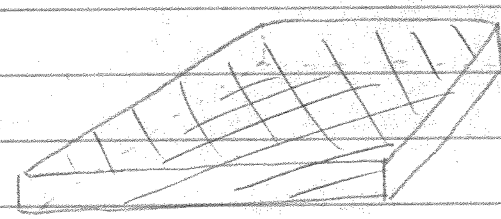
$$= \frac{1}{2} \left(\hat{R}_z(k_c) - \hat{R}_z(0) \right) \xrightarrow{k_c \to \infty} -\frac{1}{2} \hat{R}_z(0) = -\frac{1}{2} \mathrm{sgn}(m)$$

$$\therefore \sigma_H = -\mathrm{sgn}(m) \frac{1}{2} \frac{e^2}{h} \quad //$$

⑤ However, ...

1. half-Int QHE is impossible in strictly 2D system.
   c.f.) TKNN Formula

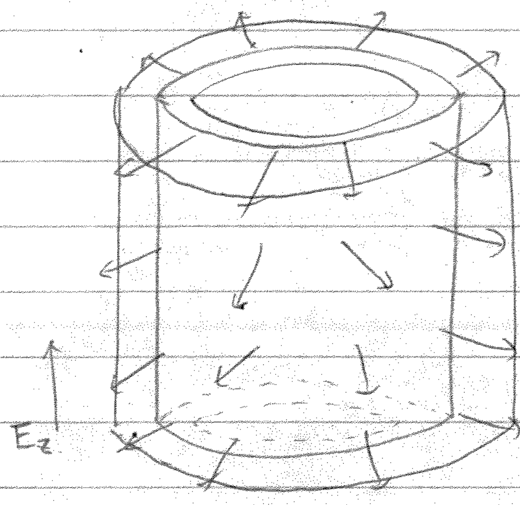
2. If you have a slab,



the top & bottom both contribute to QHE, which gives Int. QH conductance as the total

3. Symptom of 2D chiral fermions. which is impossible in real 2D sys.
   c.f.) Neilsen - Ninomiya's Theorem

IV Topological Magneto-Electric Effect



1. Ferromagnet induce magnetic field to the surface state.

$H' = m \cdot \sigma$
$= m^z \sigma^z$

2. Gap-out the surf-state
⊕ Chiral Edge Mode
⇒ QHE w/ $\sigma_{xy} = \frac{e^2}{2h}$
(Take + sgn for concreteness)
by applying E_z.

3. from Ampere's Law ($\oint_C \mathbb{B}(r) \cdot dr \propto I = \mu_0 I$),

$m_z = j_H = \frac{e^2}{2h} E_z$ c.f.) $B^z_{ind} \cdot \frac{I}{2\pi R} \cdot 2\pi R = I$

hence, finite magnetic moment appears by applying E_z

4. ME-effect only appears w/ b-TRS, as
E is even & B is odd under TRS.

V Combination of SC & TI → after studying SC