Quantum treatment

- So far, we treated the ferromagnet in mean field theory
  - This is approximate. Usually qualitatively correct but not even always that.
  - We can do better for the Heisenberg ferromagnet
- Goal: find the actual ground state and excitations

Quantum treatment

- Hamiltonian $H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$

$$= -J \sum_{\langle ij \rangle} \left(S_i^z S_j^z + \frac{1}{2} S_i^+ S_j^- + \frac{1}{2} S_i^- S_j^+ \right)$$

- Try the "obvious" ground state $|0\rangle = \prod_i |S_i^z = +S\rangle$

$$H|0\rangle = -J \sum_{\langle ij \rangle} S^2 |0\rangle = -NJS^2 \frac{z}{2} |0\rangle$$

Quantum treatment

- Is it the ground state?

$$H = -J \sum_{\langle ij \rangle} \left[\left(\frac{\mathbf{S}_i + \mathbf{S}_j}{2} \right)^2 - S(S+1) \right]$$

- Since $0 \leq |S_i + S_j| \leq 2S$ for two spins

$$E \geq -J \sum_{\langle ij \rangle} \left[\frac{2S(2S+1)}{2} - S(S+1) \right] = -JS^2 \frac{Nz}{2}$$

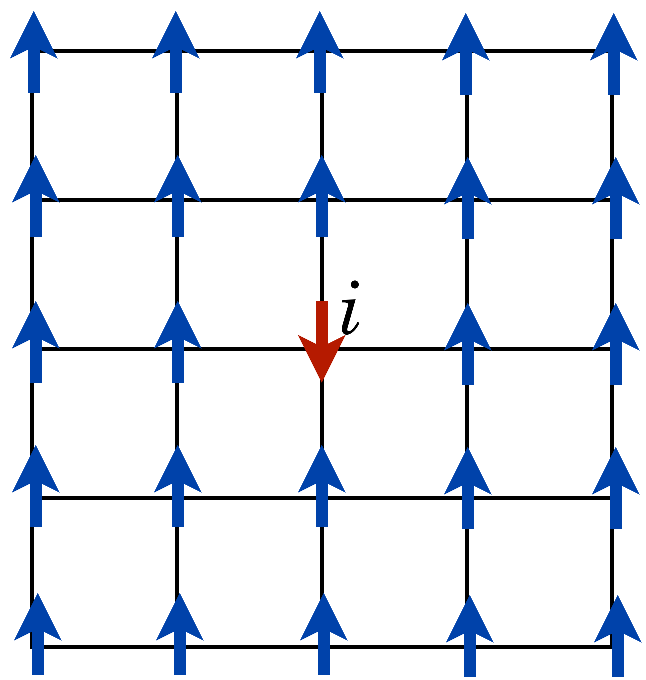
- Thus this is indeed a ground state!

Quantum treatment

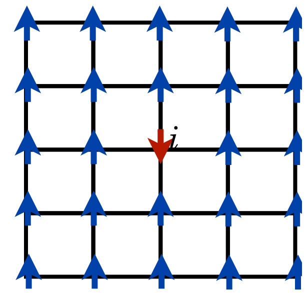
- Excitations - lower the spin once

$$|i\rangle = \frac{S_i^-}{\sqrt{2S}} \prod_j |S_j^z = S\rangle = |S_i^z = S-1\rangle \prod_{j\neq i} |S_j^z = S\rangle$$

e.g.
S=1/2



Quantum treatment



- Hamiltonian

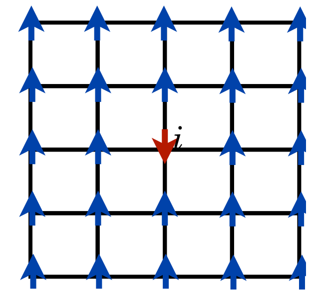
$$H = -J\sum_{\langle ij\rangle}\left(S_i^z S_j^z + \frac{1}{2}S_i^+ S_j^- + \frac{1}{2}S_i^- S_j^+\right) \equiv H_z + H_\pm$$

- zz terms

$$H_z|i\rangle = \left(E_0 - Jz[S(S-1) - S^2]\right)|i\rangle = \left(E_0 + JSz\right)|i\rangle$$

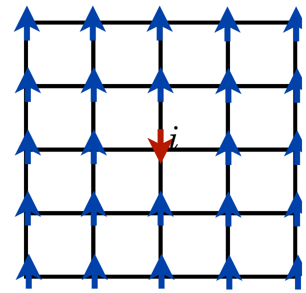
Quantum treatment

- **+- terms** $H_{\pm} = -\frac{J}{2}\sum_{\langle jk\rangle}\left(S_j^+ S_k^- + S_j^- S_k^+\right)$



$$H_{\pm}|i\rangle = -\frac{J}{2}\sum_{\langle ji\rangle} S_j^- S_i^+ |i\rangle$$
$$= -\frac{J}{2}2S\sum_{\langle ji\rangle}|j\rangle = -JS\sum_{\mu}|i+\mu\rangle$$

Quantum treatment



- All together

$$H|i\rangle = (E_0 + JSz)\,|i\rangle - JS\sum_{\mu}|i+\mu\rangle$$

- Looks like a hopping Hamiltonian
- Bloch:

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{N}}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}|i\rangle$$

$$H|\mathbf{k}\rangle = \left(E_0 + JSz - JS\sum_{\mu} e^{i\mathbf{k}\cdot\mathbf{e}_\mu}\right)|\mathbf{k}\rangle$$

$$E_0 + \epsilon(\mathbf{k})$$

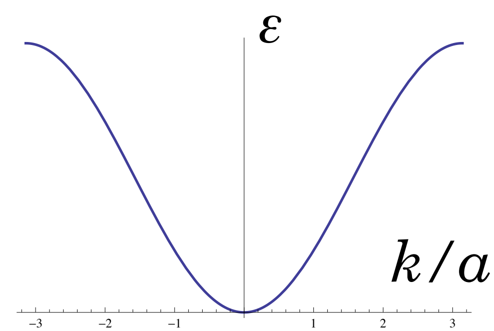
Quantum treatment

- These are “spin waves” or “magnons”

$$\epsilon(\mathbf{k}) = JSz - JS\sum_{\mu} e^{i\mathbf{k}\cdot\mathbf{e}_{\mu}}$$
$$= 2JS\left(d - \sum_{\alpha=1}^{d} \cos k_{\alpha} a\right)$$
$$\approx JSa^2k^2 \quad \text{quadratic for small k}$$

- Magnons are gapless

Magnons



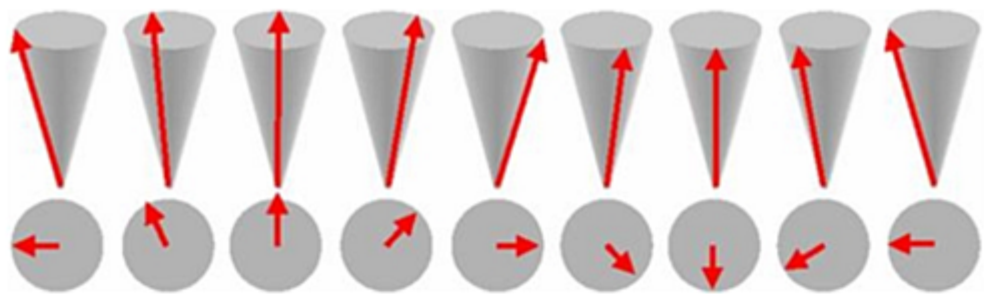
- Magnons are gapless
  - Energy vanishes as k → 0
  - This is because

$$|\mathbf{k}=0\rangle \propto S^{-}_{\mathrm{TOT}}|0\rangle$$

- This is an example of a "Goldstone mode"
  - "Goldstone's theorem": if H has a continuous symmetry that is "broken" by the ground state, there will be a gapless mode

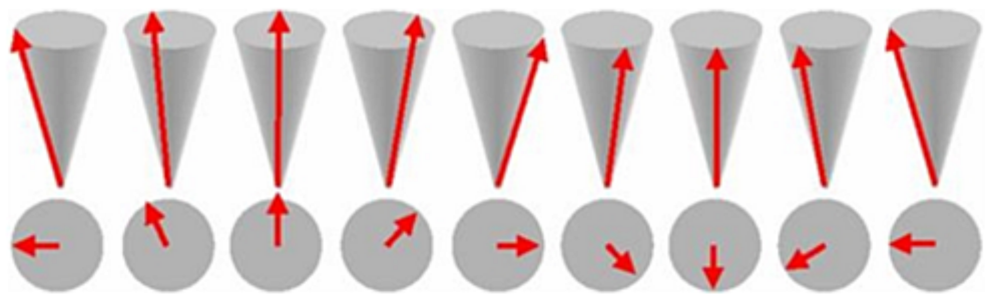
Magnons

- Spin wave/magnon can be regarded as a quantized precession wave of slightly tilted spins



- $\varepsilon \sim k^2$ behavior, a general property for ferromagnets, can be understood this way

Magnons



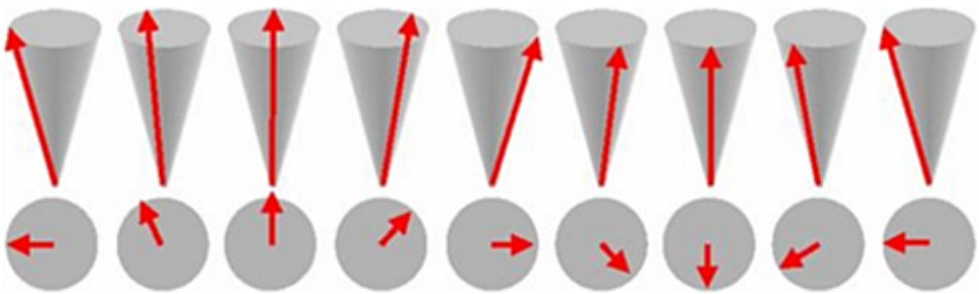
- Think of effective field due to other spins

$$\mathbf{h}(\mathbf{r}) \approx c_0 \mathbf{m} + c_1 \nabla^2 \mathbf{m} + \cdots$$

- Local spins precess in this field

$$\partial_t \mathbf{m} = \mathbf{h} \times \mathbf{m} \approx c_1 \nabla^2 \mathbf{m} \times \mathbf{m}$$

Magnons



$$\partial_t \mathbf{m} = \mathbf{h} \times \mathbf{m} \approx c_1 \nabla^2 \mathbf{m} \times \mathbf{m}$$

$$\mathbf{m} = (m_x, m_y, \sqrt{m_0^2 - m_x^2 - m_y^2}) \approx (m_x, m_y, m_0)$$

$$\partial_t \begin{pmatrix} m_x \\ m_y \end{pmatrix} = c_1 m_0 \begin{pmatrix} 0 & -k^2 \\ k^2 & 0 \end{pmatrix} \begin{pmatrix} m_x \\ m_y \end{pmatrix}$$

$$\Rightarrow \quad \omega = \pm c_1 m_0 k^2$$

Thermodynamics

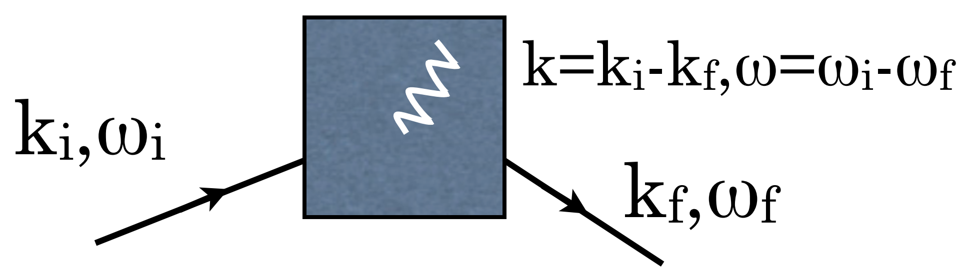
Thermodynamics of magnons is nicely discussed in Solyom, p.527-529

Neutron scattering

- Neutron has a S=1/2 similar to an electron, and its own dipole moment, which interacts with magnetic dipoles in materials

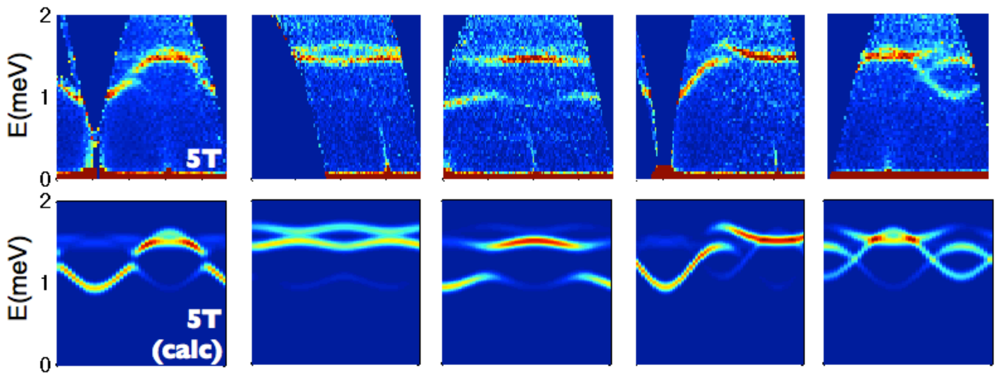
$$H_{d-d} = -\frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}\cdot\mathbf{r})(\mathbf{m}'\cdot\mathbf{r}) - \mathbf{m}\cdot\mathbf{m}'\right]$$

- Consequently, a neutron can exchange energy and momentum with electronic spins

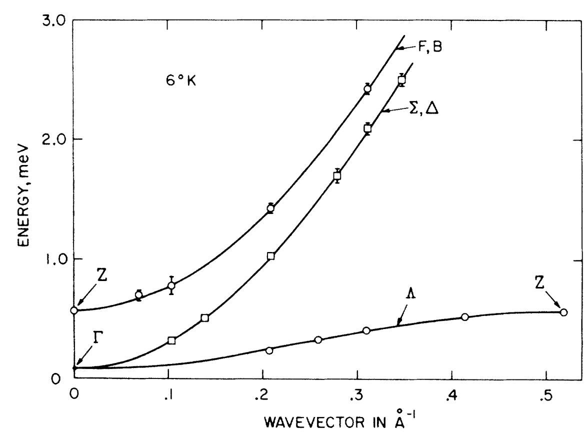


Neutron scattering

- Integrate over all energies: get total scattering
  - tool for detecting magnetic ordering in solids
- Resolve both momentum and energy of neutrons: measure spin waves



example of spin waves in $Yb_2Ti_2O_7$ (2012)



first measurement in an insulating FM: $CrBr_3$, 1971