

Physics 23

Assignment 1 Solutions

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(Dated: October 6, 2006)

E18-2

a)

$$f = \frac{\text{number of oscillations}}{\text{total time for oscillations}} = \frac{12}{30s} = 0.4s^{-1} \quad (1)$$

b)

$$v = \frac{\text{distance traveled by wave}}{\text{time}} = \frac{15m}{5s} = 3m/s \quad (2)$$

c)

$$\lambda = \frac{v}{f} = \frac{5m/s}{0.4s^{-1}} = 7.5m \quad (3)$$

E18-10

The velocity of the wave is given by (in this case, the phase velocity is the same as the group velocity of the wave, for those of you who may know Fourier analysis and linear algebra)

$$v = \frac{\omega}{k} = \frac{317 \text{ rad/s}}{23.8 \text{ rad/m}} \approx 13.32m/s. \quad (4)$$

We know that the mass density of the string is related to velocity and tension by

$$\mu = \frac{T}{v^2} \approx \frac{16.3 \text{ N}}{13.32m/s^2} \approx 0.0919kg/m. \quad (5)$$

E18-16

a) As the cylindrical wave travels outward, it spreads its power over increasingly large area since the area of the side of a cylinder is proportional to the radius. Since the medium is absorbs no energy, the total power in the wave must remain constant as it propagates. Therefore the power per unit area of the wave, namely the intensity, has to decrease with radius. We then have

$$I = \frac{P_{av}}{A} \sim \frac{1}{r} \quad (6)$$

so that the total average power stays constant and radius-independent. Here A is the area of the cylinder with radius r that the wave is passing through.

b) We know that $I \sim y_m^2$, where y_m is the amplitude. (For multi-dimensional waves, this is actually not completely obvious to show) Therefore we have

$$y_m \sim I^{1/2} \sim \frac{1}{r^{1/2}}. \quad (7)$$

Alternatively, we can write down the 3-D wave equation in cylindrical coordinates and obtain the solution explicitly. If interested, we can do so in discussion or office hours.

E18-20

We have two waves with non-equal amplitudes adding together, so to begin with we literally write down the resultant wave as the sum of the two pieces. To simplify algebra, we choose to define our observation point to be the origin so that $kx = 0$ for both waves. The combined wave can be written as (you can choose to use cosine instead)

$$y(x = 0, t) = y_{m1} \sin(\omega t + \pi/2) + y_{m2} \sin(\omega t) = y_{m1} \cos(\omega t) + y_{m2} \sin(\omega t), \quad (8)$$

where ω is the common angular frequency of the waves and we used the fact that sine and cosine are $\pi/2$ radian out of phase relative to each other. Next, we rewrite y_{m1} and y_{m2} in the form

$$\begin{aligned} y_{m1} &\equiv y_m \sin(\theta) \\ y_{m2} &\equiv y_m \cos(\theta). \end{aligned} \quad (9)$$

Here θ and y_m are both yet unspecified real parameters. Inserting Eqn.(9) into Eqn.(8) and applying a well known trigonometric identity (the sum angle formula), the combined wave turns magically into

$$y(x = 0, t) = y_m \sin(\omega t + \theta). \quad (10)$$

Here comes the critical step, we know that $\sin^2(\theta) + \cos^2(\theta) = 1$, so Eqn.(9) implies

$$\left(\frac{y_{m1}}{y_m}\right)^2 + \left(\frac{y_{m2}}{y_m}\right)^2 = 1. \quad (11)$$

We then solve for y_m to obtain

$$y_m^2 = y_{m1}^2 + y_{m2}^2, \quad (12)$$

which is the desired result if we identify y_m as the amplitude of the resulting wave. We now also see that θ is actually the overall phase shift of the new wave. Using numbers given in the problem, we find that $y_m \approx 5.27\text{cm}$.

The solution presented here is very tedious and relies almost exclusively on remembering or looking up (took me a lot of time) the relevant identity. If you have a faster and hopefully more physical approach, please enlighten me with the knowledge.

E18-30

a) The violin string vibrations excite a compression wave of air molecules at the same frequency. Therefore the emitted sound wave has

$$f = \frac{nv}{2L} \approx \frac{(1)(250\text{m/s})}{(2)(0.15\text{m})} \approx 833\text{Hz}. \quad (13)$$

b) Since the speed of sound is given, the wavelength is simply

$$\lambda = \frac{v}{f} \approx \frac{348\text{m/s}}{833\text{s}^{-1}} \approx 0.418\text{m} \quad (14)$$

Note that human ears are sensitive only to the frequency of sound, so the much longer wavelength does not affect our hearing of the instrument.

P18-4

a)

$$f = \frac{\omega}{2\pi} = \frac{4.08\text{rad/s}}{2\pi \text{ rad}} \approx 0.65\text{Hz} \quad (15)$$

b)

$$\lambda = \frac{v}{f} \approx \frac{82.6\text{cm}}{0.65\text{s}^{-1}} \approx 1.27\text{m} \quad (16)$$

c) The wave number is

$$k = \frac{2\pi}{\lambda} \approx \frac{2\pi\text{rad}}{1.27\text{m}} \approx 4.95\text{rad/m}. \quad (17)$$

Next, from the wave function given by the problem, we know that

$$k(9.60\text{cm}) + \phi = 1.16\text{rad} \quad (18)$$

where ϕ is the phase shift of the wave, which is determined by using the value of k just found. Putting everything together into the standard form of a traveling transverse wave, we have

$$y(x, t) = (5.12\text{cm})\sin[(4.95\text{rad/s})x - (4.08\text{rad/s})t + 0.685\text{rad}] \quad (19)$$

d) The tension is

$$T = \mu v^2 \approx (3.86\text{g/cm})(82.6\text{cm/s})^2 \approx 0.263\text{N}. \quad (20)$$

P18-8

a) Let's choose the bottom of the rope to be the origin $y = 0$. Then at any height y on the rope, the upward tension force must balance the weight of the portion of the rope so that the rope does not break. Therefore we have

$$T(y) = \frac{m}{L}yg \quad (21)$$

since the rope is of uniform density m/L and thus $\frac{m}{L}y$ is the mass of the rope below height y . The speed of a traveling wave on the rope is then obtained as usual by

$$v = \left(\frac{T}{\mu}\right)^{1/2} = \left(\frac{mgy/L}{m/L}\right)^{1/2} = (gy)^{1/2}. \quad (22)$$

b) Eqn.(22) can be rewritten as

$$v = \frac{dy}{dt} = (gy)^{1/2}. \quad (23)$$

We can rearrange the above expression to solve for dt and then integrate both sides to find

$$\int_0^\tau dt = \int_0^L \frac{dy}{(gy)^{1/2}}. \quad (24)$$

Here τ is the total time it takes a wave pulse to travel the length of the rope from 0 to L .

The integral yields

$$\tau = 2 \left(\frac{L}{g}\right)^{1/2} \quad (25)$$

as desired.

c) As demonstrated above, the mass of the rope has no effect on the results of a) and b).

P18-22

a) The standing waves on the two wires share the same frequency because they are connected and excited by a single external source. However, the wavelength and speed of the waves on the two wires need NOT to be the same since they have different densities. We therefore have two separate standing waves that have a common frequency and share a common boundary (where they are connected) but oscillate at different modes, or with different n 's. The mode

numbers n_1 and n_2 of the two wires (1 for aluminum and 2 for steel) must be integers and so their ratio must be a rational number. Furthermore, that ratio is fixed by the given parameters in the problem although n_1 and n_2 individually are not. We have

$$f = \frac{nv}{2L} \quad (26)$$

for any standing wave on a string of uniform density. Therefore

$$\frac{n_2}{n_1} = \frac{2L_2v_2f}{2L_1v_1f}. \quad (27)$$

Since $v = (T/\mu)^{1/2}$ and the tension common to both strings is provided by the mass m , Eqn.(27) simplifies to

$$\frac{n_2}{n_1} = \frac{L_2}{L_1} \left(\frac{\mu_2}{\mu_1} \right)^{1/2} \approx 2.5 \approx \frac{5}{2} \quad (28)$$

when we put in the numerical values given by the problem. Note that here we can use the volume mass densities of the materials as their linear mass densities because the wires have the same cross-sectional area and are under the same tension, so that the area cancels out from the ratio. We see that the smallest integer ratio of mode numbers is $5/2$, so that the lowest possible mode on the aluminum wire that results in the junction being a node (or a boundary to both waves) is $n_1 = 2$. The corresponding frequency, common to both wires, is

$$f = \frac{n_1v_1}{2L_1} = \frac{n_1}{2L_1} \left(\frac{T}{\mu_1} \right)^{1/2} = \frac{n_1}{2L_1} \left(\frac{mg}{\rho_1 A} \right)^{1/2} \approx 323Hz \quad (29)$$

when we put in numbers given by the problem, where ρ_1 is the density and A is the cross-sectional area of the wire. Here we have to include the area because we are no longer considering the ratio between two densities. (recall that for a three dimensional object, the linear mass density along a particular direction is simply the volume mass density multiplied by the cross-section area perpendicular to the direction of interest)

b) There is one node ($n=2$) in the aluminum wire and four nodes ($n=5$) in the steel wire. (remember n is the number of anti-nodes in the wave) However the shared junction now also count as a node since its on the length of the combined wave, so there are six modes in total.