Physics 23

Assignment 4 Solutions

(50 Points)

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E26-8 (4 Points)

If the charges on the line x = a where +q and ..q instead of +2q and ..2q then at the center of the square E = 0 by symmetry. This simplifies the problem into finding E for a charge +q at (a, 0) and -q at (a, a). This is a dipole, and the field is

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{[x^2 + (d/2)^2]^{3/2}}, \tag{1}$$

where **p** is the dipole moment, x is the perpendicular distance from the center of the dipole and d is the distance between the charges. Here we have $d = a$, $\mid \mathbf{p} \mid = qa$ and $x = 2a$. Putting in the given numbers, we obtain$\mid \mathbf{E} \mid \approx 1.11 \times 10^5 N/C$.

E26-16 (6 Points)

The horizontal component of the net electric field at point P is zero because the contributions from all charges cancel. Therefore the net electric field only has a vertical component that points downward. The magnitude of the electric field is then

$$E = \frac{1}{4\pi\epsilon_0}\int_0^{\pi/2}\frac{\lambda sin(\theta)d\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r^2} = \frac{q}{\pi^2\epsilon_0 r^2}. \tag{2}$$

Here θ is the normal polar angle coordinate and we used $\lambda = 2q/(\pi r)$.

E26-22

The picture should look exactly like the field of two equal but opposite point charges except the vector field lines going into the -2q charge should be twice as long as those coming immediately out of the +q charge.

E26-36

a) (2 Points)

$$p = qd \approx 9.22 \times 10^{-15} C \cdot m \tag{3}$$

b) (4 Points) The potential energy of a dipole in an uniform electric field is

$$U = \mathbf{p} \cdot \mathbf{E} = pEcos(\theta), \tag{4}$$

where θ is the angle between the axis of the dipole and the electric field. By our convention, $\theta = 0$ and $\theta = \pi$ when the dipole is parallel and anti-parallel to the electric field, respectively. The change in potential energy between the two orientations is therefore

$$\Delta U = pE - (-pE) = 2pE \approx 2.03 \times 10^{-11} J. \tag{5}$$

P26-4 (8 Points)

As shown in class, we first obtain a general expression for the electric at point P due to the four charges. The field is entirely in the y-direction and its magnitude is

$$E = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(z+d)^2} + \frac{q}{(z+d)^2} - \frac{2q}{z^2}\right]. \tag{6}$$

We now take the point P to be very far away from the charges ($d << z$) and Taylor expand terms containing $z \pm d$ in powers of the dimension-less small parameter d/z. By first factoring out a $1/z^2$ from every term, we obtain

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\frac{1}{[1+(d/z)]^2} + \frac{1}{[1-(d/z)]^2} - 2\right]. \tag{7}$$

We expand about $d/z = 0$ since if we go to infinitely far away, we can no longer see the finite separation between the charges. We thus have

$$\frac{1}{[1 \pm (d/z)]^2} = 1 \mp \frac{2d}{z} + \frac{3d^2}{z^2} + O(\left(\frac{d}{z}\right)^4). \tag{8}$$

We ignore terms of order $(d/z)^4$ and higher since they are all vanishingly small compared to the first two terms at very far away. The electric field at point P is then approximately

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[1 - \frac{2d}{z} + \frac{3d^2}{z^2} + 1 + \frac{2d}{z} + \frac{3d^2}{z^2} - 2\right] + higher\ order\ terms. \tag{9}$$

Using $Q \equiv 2qd^2$, we obtain the desired result:

$$E \approx \frac{1}{4\pi\epsilon_0}\frac{3Q}{z^4}. \tag{10}$$

Note that the terms of order d^2/z^2 cancelled out. Physically, this means that if we go very far away from the charges, we can no longer resolve the configuration in sufficient detail to tell that it is made of two dipoles. Thus the dipole-like contribution to the field cancels and we see only the quadrupole field to a very good approximation.

P26-6 (8 Points)

The vertical component of the electric field at point P is simply half of the field of a infinitely long rod (in both directions). Defining the vertical direction as the y-axis, the magnitude of the y-component of the electric field is (see Eqn.26-17 of the textbook)

$$E_y = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{R}, \tag{11}$$

since R is the vertical distance from the rod to point P. In the horizontal direction, we pick out the x-component of the of the field and integrate across all the charge on the rod from 0 to $+\infty$. Therefore we have

$$E_x = \int dE_x = \int dEsin(\theta) = \frac{1}{4\pi\epsilon_0}\int_0^\infty \frac{\lambda dx}{R^2+x^2}\frac{x}{(R^2+x^2)^{1/2}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{R}. \tag{12}$$

Here θ is the angle between a point on the rod and the line connecting that point to P. We see that E_x and E_y are equal in magnitude and thus must make an angle of $\pi/4$ radian with the rod. Furthermore, this result holds regardless of the value of R.

P26-8 (10 Points)

First, note that the net electric field at the center is entirely vertical (in z-direction) since the horizontal components all cancel by symmetry.

Referring to the attached figure, we see that the charge dq carried by a thin ring of height dz is

$$dq = 2\pi r\sigma dz = 2\pi r dz\frac{q}{2\pi R^2} = \frac{qrdz}{R^2} \tag{13}$$

The distance between each ring and the observation point is simply R. Therefore the electric field is

$$E_z = \int dEcos(\theta) = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{qrdz}{R^2}\frac{1}{R^2}\frac{z}{R}. \tag{14}$$

Next, we use $r^2+z^2=R^2$ to rewrite r in terms of z. We have

$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{q(R^2-z^2)^{1/2}dz}{R^2}\frac{1}{R^2}\frac{z}{R} = \frac{q}{4\pi\epsilon_0}\int_0^R \frac{dz(R^2-z^2)^{1/2}z}{R^5}. \tag{15}$$

The final result is

$$\mathbf{E} = \frac{q\hat{\mathbf{j}}}{12\pi\epsilon_0 R^2} \tag{16}$$

P26-14 (8 Points)

The electric field applies equal but opposite forces on the two charges that are at two ends of the dipole and thus applies a net torque (into or out of the page depending on how you draw the picture.) on the dipole. According to the attached figure, we have

$$\mid \tau \mid = \mid \mathbf{r} \times \mathbf{F} \mid = \mid \mathbf{r} \mid\mid \mathbf{F} \mid sin(\theta) = I\frac{d^2\theta}{dt^2}. \tag{17}$$

Here **r** is the vector from where we wish to calculate the torque about to where the force causing the torque is applied, **F** is the applied force and I is the moment of inertia of the dipole. The most sensible choice of **r** is from the center of the dipole (axis of rotation) to one of the charges. For small oscillations, $sin(\theta) \approx \theta$. We therefore have

$$\frac{d^2\theta}{dt^2} \approx \frac{2qE}{I}\frac{d}{2}\theta = \frac{pE}{I}\theta, \tag{18}$$

where p is the dipole moment. We thus have the equation of motion of a simple harmonic oscillator with angular frequency $\omega^2 = pE/I$.