Physics 23
Assignment 6 Solutions
(48 Points)

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E28-6 (6 Points)

Electro-static force is conservative and is the only force acting on the electron, thus energy is conserved. We have

$$\Delta U + \Delta K = 0 = (U_f - U_i) + (K_f - K_i) = e\Delta V + (0 - mv_i^2/2) \tag{1}$$

because the change in potential energy is due entirely due to the potential difference between the plates and the final kinetic energy is 0. We solve to obtain $v_i \approx 6.0 \times 10^7 m/s$.

E28-12 (8 Points)

a) (4 Points) Let $r_1 = 15cm$ and $r_2 = 5cm$. Since the electric potential is a scalar, we simply add the contributions from each charge for each point. With the usual convention of setting $V(\infty) = 0$, we have for points A and B

$$V_a = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} \approx 6 \times 10^4 V \tag{2}$$

and

$$V_b = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_2} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_1} \approx -7.8 \times 10^5 V. \tag{3}$$

b) (2 Points) The work done to push q_3 from B to A is path independent and therefore only depends on the potential difference between the points. We then have

$$W = q_3\Delta_{AB} = -q_3(V_A - V_B) \approx 2.5J. \tag{4}$$

c) (2 Points) The work done on q_3 went into electro-static potential energy stored in the electric field of the final charge configuration.

E28-20

The potential at the surface grain depends on the total charge on its surface. With V=0 at infinity, the potential at the surface is

$$V(R = 0.1\mu m) = \frac{1}{4\pi\epsilon_0}\frac{ne}{R} = -400V, \tag{5}$$

where n is the number of electrons. Putting in the numbers, we have $n \approx 2.8 \times 10^5$.

E28-30 (8 Points)

a) (2 Points) The greatest absolute value of the electric occurs where the space derivative of the potential is the greatest since

$$\mathbf{E} = -\nabla V. \tag{6}$$

In our case the problem is one dimensional so we simply have

$$|E| = \left|\frac{\partial V}{\partial x}\right|. \tag{7}$$

Since the potential is a linear function of x, we see that the electric field is the greatest where we have the steepest slope, which occurs between d and e.

b) (2 Points) The electric field is the smallest where the slope of V(x) is the least steep, which happens between b, c and e, f.

c) (4 Points)

E28-42 (6 Points)

a) (2 Points) The spheres become one conducting surface upon connecting. Charges are free to flow in a conductor, so the initial potential difference between the spheres will cause positive charges to flow from higher to lower potential (or vice versa if the charges are negative) so that they can minimize their total energy. The process will stop when the two spheres reach the same potential since at that time the electric field in the system will be 0 (recall that electric field is the space derivative of the potential and thus is non-zero only when there is a non-zero potential difference) and there will no longer be an electro-static force acting on the charges. This simple argument explains why the electric field inside a conductor is 0 and the surface of a conductor is an equi-potential. In conclusion, $V_1 = V_2$ after the system reaches equilibrium.

b) (4 Points) The total charge is conserved so after the system reaches equilibrium we must have

$$q = q_1 + q_2. \tag{8}$$

The potential on the surface of each sphere is equal, so we have (again setting V=0 at infinity)

$$V_1 = V_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2}, \tag{9}$$

where r_1 and r_2 are the radii of each sphere and $r_1 = 2r_2$. Solving the system of equations for q_1 and q_2, we obtain $q_1 = q/3$ and $q_2 = 2q/3$.

P28-4 (12 Points)

a) (4 Points) We start by writing down the potential difference between the center of the sphere and a point at distance r from the center:

$$\Delta V = V(r) - V(0) = -\int_0^r \mathbf{E} \cdot \mathbf{dr}', \tag{10}$$

where $\mathbf{dr}'$ is a small displacement vector pointing radially outward. Since $\mathbf{E}$ and $\mathbf{dr}'$ are either parallel or anti-parallel (depending on sign of q) and we set V(0)=0, Eqn.(10) yields the result via direct integration,

$$V(r) = -\int_0^r E dr' = -\int_0^r \frac{1}{4\pi\epsilon_0}\frac{qr'}{R^3}dr' = -\frac{1}{8\pi\epsilon_0}\frac{qr^2}{R^3}. \tag{11}$$

b) (2 Points) Since our V(r) is also the potential difference between the center and a point a distance r from the center, we find the potential difference between the center and the surface simply by setting r= R in Eqn.(11). Therefore

$$\Delta V_{R,0} = -\frac{1}{8\pi\epsilon_0}\frac{q}{R}. \tag{12}$$

Note that the potential difference between the center and the surface is NOT equal to that of between the surface and infinity.

c) (6 Points) We cannot integrate the electric field from r to infinity directly because the functional forms of the field in and outside of the sphere are different. Thus we break the problem into two parts. First, calculate the potential difference between the surface and infinity with $V(\infty) = 0$:

$$\Delta V_{\infty,R} = V(\infty) - V(R) = -\int_R^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}dr', \tag{13}$$

where we now use the electric field OUTSIDE of the sphere and dropped the vector notation for the same reason as before. Since $V(\infty) = 0$, we can extract V(R) by keeping careful track of signs:

$$V(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}. \tag{14}$$

Next, we compute the potential difference between some distance r inside the sphere and the surface:

$$\Delta V_{R,r} = V(R) - V(r) = -\int_r^R \frac{1}{4\pi\epsilon_0}\frac{qr'}{R^3}dr' = -\frac{1}{8\pi\epsilon_0}\frac{q(R^2 - r^2)}{R^3}. \tag{15}$$

Note that here we used the electric field INSIDE the sphere. To extract V(r) with the boundary condition $V(\infty) = 0$, we use in Eqn.(15) the V(R) just obtained with the same boundary condition. The result is

$$V(r) = \frac{1}{8\pi\epsilon_0}\frac{q(R^2 - r^2)}{R^3} + \frac{1}{4\pi\epsilon_0}\frac{q}{R}, \tag{16}$$

which agrees with the textbook once simplified.

P28-6

a) We have another conservation of energy problem. The change in electro-static potential energy (particle comes in from infinity with the potential being 0 at infinity) plus the change in kinetic energy must be 0. We then have

$$0 = \Delta U + \Delta K = \left(\frac{1}{4\pi\epsilon_0}\frac{qQ}{d} - 0\right) + (0 - K) \tag{17}$$

and therefore

$$d = \frac{1}{4\pi\epsilon_0}\frac{qQ}{K}. \tag{18}$$

b) Again using energy conservation, we obtain

$$K = \frac{1}{4\pi\epsilon_0}\frac{qQ}{2d} + \frac{1}{2}mv^2. \tag{19}$$

Using the result of d from part a) (since d is the distance at which the particle would be instantaneously stopped), we obtain $v = (K/m)^{1/2}$.

P28-10 (10 Points)

The general expression for the electric potential due to a continuous surface charge distribution is (again with V=0 at infinity)

$$V(\mathbf{r}) = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{\mid \mathbf{r} - \mathbf{r}' \mid} = \int_{A'} \frac{1}{4\pi\epsilon_0} \frac{\sigma(\mathbf{r}')dA'}{\mid \mathbf{r} - \mathbf{r}' \mid}, \tag{20}$$

where $\mathbf{r}$ is the vector from the origin (of some coordinate system that we define when starting the problem) to the observation point, $\mathbf{r}'$ is the vector from the origin to a small piece of charge on the distribution, (thus subtracting them gives the vector from a small piece of charge to the observation point) and A' is the area of surface charge distribution to be integrated over.

We adopt polar coordinates with the origin at the center of the annulus, in which $\mathbf{r} = 0$ (since we want the potential at the center) and dA' is simply the usual polar coordinate area element. Eqn.(20) then becomes

$$V(r=0) = \int_a^b \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{\sigma(r')r'dr'd\theta'}{r'} = \frac{1}{4\pi\epsilon_0} \int_a^b \sigma(r')dr' \int_0^{2\pi} d\theta'. \tag{21}$$

Here we integrated across the area of the annulus in order to sum up contributions to the potential from all charges on the surface. The angular integral simply gives 2π and we can perform the radial integral by putting in the given radius-dependent charge density $\sigma = k/r'^3$. The result is

$$V(r=0) = \frac{k}{4\epsilon_0} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \tag{22}$$

The total charge in the annulus is given by a similar calculation:

$$Q = \int_{A'} \sigma(r')dA' = \int_a^b \int_0^{2\pi} \frac{k}{r'^3} r'dr'd\theta' = 2\pi k \frac{b-a}{ab}. \tag{23}$$

Now solve for k in Eqn.(23) and substitute into Eqn.(22) to obtain the desired result in terms of Q:

$$V(r=0) = \frac{Q}{8\pi\epsilon_0} \frac{a+b}{ab}. \tag{24}$$