Week 1

1 The relativistic point particle

Reading material from the books

- *Zwiebach, Chapter 5 and chapter 11*
- *Polchinski, Chapter 1*
- *Becker, Becker, Schwartz, Chapter 2*

1.1 Classical dynamics

The first thing we need to understand is what the configuration space of a point particle in four dimensional spacetime looks like (any dimension will do, so long as we insist on Lorentz symmetry).

At time t, we should be able to identify the location of the particle by three coordinates $x^i(t)$ and this should be all of it.

Notice that although this is a convenient way to parametrize the trajectory of the particle, it is rather hard to pass from one coordinate system to another, because under relativistic transformations the x^i mix with t and writing this new information in terms of t' is in general hard.

A much more convenient point of view is to understand things from a more geometrical perspective: The particle motion sweeps a one dimensional curve in four dimensional spacetime.

We can parametrize this trajectory by four functions of an auxiliary variable τ:

$$x^\mu(\tau)$$

Now it is easy to do Lorentz transformations, because we can just act linearly on the corodinates x^μ.

This choice makes the Lorentz symmetry more manifest. However, we have gone from three functions of t to four functions of τ. It would seem that in this way we are making the configuration space bigger. Clearly, we could choose another parametrization of the curve, in terms of $\tau'(\tau)$, and we would be describing the same curve in different "internal coordinates".

It should be the case that the dynamics does not care for which "internal coordinate" system we use. Therefore the dynamical principle should be invariant under reparametrizations of τ.

We usually formulate dynamics by a least action principle. We want this action principle to be relativistically invariant (thus making Lorentz symmetry manifest).

We should also make a null hypothyesis: only the geometric embedding of the curve in spacetime matters to define the action. (We mean by null hypothesis here that we make minimal additional input on the theory: we only use the natural geometric invariants to define our dynamics).

Finally, we would want the wordline description to be local on the embedding:

We should be able to formulate the action principle as

$$S \sim \int d\tau \mathcal{L}(x^\mu(\tau), \dot{x}^\mu(\tau))$$

and possibly higher derivatives with respect to τ.

Making the geometric hypothesis for the dynamics is simple:

In an euclidean geometry, we would assign the total action to the total length of the curve. This is geometric and local: each little piece of curve has a length, and the total length is computed by adding these pieces together.

In a Lorentzian geometry we have a similar quantity: the proper time along the curve (take a clock along with the particle motion, and measure how much time passes in that clock).

For infinitesimal displacements we have

$$-ds^2 = (x^\mu(\tau + \delta\tau) - x^\mu(\tau))^2 \tag{1}$$

(In the square above we contract indices using the Lorentzian constant metric. We use metric with (-,++++++) signature).

It is natural to consider the following action:

$$S \sim \int ds \tag{2}$$

Because S has units of Energy times time (dimensionless in natural units), we need to add dimensionful quantities to S to make it match:

$$S = -m \int ds = -m \int \frac{ds}{d\tau} d\tau = -m \int \sqrt{-\dot{x}^\mu(\tau)^2} d\tau \tag{3}$$

Where m is called the mass of the particle. It is forced on us by dimensional analysis.

Notice that this action is reparametrization invariant: we can change τ to $\tau'(\tau)$ and it would take the same form in the new coordinate system. This is a symmetry of the problem that we can use to define the time coordinate however we want. In particular we can choose $t = \tau$

In this form the action takes the form

$$S = -m \int \sqrt{1 - \vec{v}^2} dt \tag{4}$$

and we can expand it around $\vec{v} = 0$, so that we get the classical non-relativistic point particle Lagrangian $\mathcal{L} \sim \text{Const} + \frac{1}{2} m \vec{v}^2 + \dots$.

The equations of motion are

$$\frac{\partial}{\partial t} p_i = 0 \tag{5}$$

where

$$p_i = \frac{\partial L}{\partial v^i} = m \frac{v^i}{\sqrt{1 - \vec{v}^2}}$$

is the relativistic momentum.

Similarly, we can find the Energy (Hamiltonian) by a Legendre transform. We use

$$\vec{v}^2 = \frac{p^2}{p^2 + m^2}$$

, so that

$$\begin{aligned} H &= p_i v^i - \mathcal{L} = \frac{m \vec{v}^2}{\sqrt{1 - \vec{v}^2}} + m \sqrt{1 - v^2} && (6) \\ &= \frac{m}{\sqrt{1 - v^2}} = \frac{m}{\sqrt{m^2 / (p^2 + m^2)}} && (7) \\ &= \sqrt{p^2 + m^2} && (8) \end{aligned}$$

This is the familiar energy momentum-relation for a relativistic particle.

It is easy to solve the dynamics: $\dot{p}^i = 0$, that one can convert to $\dot{v}^i = 0$, so the particles move in straight lines.

Caution: The quantization is tricky:

We can take p_i and replace it by derivatives w.r.t. x^i, but the Schrodinger equation is non-local in space-time anymore,

$$H \sim \sqrt{m^2 - \partial_x^2}$$

because we depend on square roots of differential operators (expanded in Taylor series we need derivatives of a function to all orders).

One also runs into trouble trying to make sense of Lorentz tranformations whereby ∂_t mixes with the ∂_x.

2 Lightcone

We want to consider the same system as before, but we are going to choose our time coordinate in a slightly less obvious way.

The idea is to do a $2 + (D - 2)$ split of the coordinates. $x^{0,1}$ are singled out, and we have the rest of the coordinates as $x_\perp$

Now, define

$$x^\pm = x^0 \pm x^1$$

(Note: in Polchinski's book, the lightcone coordinates are chosen with an extra $1/\sqrt{2}$. This makes it similar to a rotation by $\pi/4$ in Euclidean coords. The reader can do the same calculations as noted here with these other conventions.)

In the new coordinates, the metric tensor is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dx^+ dx^- + dx_\perp^2 \tag{9}$$

(In these conventions $g_{+-} = g_{-+} = -1/2$, so that the inverse metric has $g^{+-} = g^{-+} = -2$).

Again, we can start with the action (2), and we follow the same procedure as before, but we choose a different time running on the lightcone: $x^+ = \tau$.

In these coordinates, the action becomes

$$S = -m \int \sqrt{\frac{dx^-}{d\tau} - \left(\frac{dx_\perp}{d\tau}\right)^2} \tag{10}$$

We get then that the canonical conjugate momenta are given by

$$p_\perp = m\frac{\dot{x}_\perp}{\sqrt{\dot{x}^- - \dot{x}_\perp^2}} \tag{11}$$

$$p_- = -\frac{m}{2\sqrt{\dot{x}^- - \dot{x}_\perp^2}} = -p^+/2 \tag{12}$$

Notice that in the definition above the quantity p^+ is bigger than zero. This is, the choice of sign in the square root is positive.

Now the lightcone energy is the canonical conjugate to τ. It is the Legendre transform of the Lagrangian above.

To find the Legendre transform, we use the last equation to eliminate the square roots that appear in various places.

$$p_\perp \dot{x}^\perp = p_\perp^2 \frac{\sqrt{\dot{x}^- - \dot{x}_\perp^2}}{m} \tag{13}$$

$$= \frac{p_\perp^2}{2m \cdot (-p_-)/m} = \frac{p_\perp^2}{p^+} \tag{14}$$

Similarly, we find that

$$p_- \dot{x}^- = p_-((\dot{x}_\perp)^2 + m^2/4p_-^2) = p_- p_\perp^2 \left(\sqrt{\dot{x}^- - \dot{x}_\perp^2}^{\,2}\right)/m^2 + m^2/4p_- \tag{15}$$

$$= \frac{{p_\perp}^2}{4p_-} + \frac{m^2}{4p_-} \tag{16}$$

While $-\mathcal{L} = -m^2/(2p_-)$ Putting it all together, we get that

$$H = p_\tau = p_+ = -\frac{p_\perp^2}{4p_-} - \frac{m^2}{4p_-} = \frac{p_\perp^2}{2p^+} + \frac{m^2}{2p^+} \tag{17}$$

The equation of motion of x^- tells us that $\dot{p}_- = 0$. If we impose this equation above, the Hamiltonian looks like a non-relativistic Hamiltonian where p^+ plays the role of mass. In this way, we have eliminated the square roots of the hamiltonian, and we recover a local Schrodinger equation in the (transverse) lightcone. Notice that the equation above is algebraic in the transverse coordinates to the lightcone.

This is straightforward to quantize for the transverse coordinates (we keep the total lightcone momentum $p^+ \sim p_-$ fixed, and introduce standard commutators for $p_\perp$ and $x_\perp$. We also require that p_- is the canonical conjugate operator to x^-.

Notice, moreover, that at this stage one can take the limit $m \to 0$ in a harmless way, whereas the limit $m \to 0$ of the standard point particle action is sick.

3 Covariant approach to point-particle

The main reason formulas before look cumbersome is that we had a non-covariant split between space and time. Also, the lagrangian had a square root that made life complicated.

It is convenient to introduce a Lagrange multiplier η to get rid of the square root:

$$S = -\frac{1}{2} \int d\tau \left[\eta^{-1} \left(\frac{dx^\mu}{d\tau} \right)^2 - m^2 \eta \right] \tag{18}$$

It is easy to show that eliminating η reintroduces the square root.

The equation of motion of η is

$$-\eta^{-2} \left(\frac{dx^\mu}{d\tau} \right)^2 - m^2 = 0 \tag{19}$$

So

$$\eta = m^{-1} \sqrt{ - \left(\frac{dx^\mu}{d\tau} \right)^2 } \tag{20}$$

And we recover the original point particle action that we began with. The actions are equivalent in that the solutions of the equations of motion are the same.

Notice also that for the action above, withη included, we can take the limit $m \to 0$ without trouble: the limit of a mssless particle is smooth (just like in the lightcone point particle). Then, in that limit, we can not eliminate the field η by solving for η in terms of it's equation of motion.

At first sight it might seem that the action 18 is not reparametrization invariant anymore: this is solved by transforming η as a tensor, so that $\eta d\tau = \eta' d\tau'$ is invariant.

The reparametrization invariance of the action should be understood as a gauge symmetry: a redundancy of the description, not as a symmetry that relates different solutions of the "theory"

- **Interpretation of η:**

  If we had a worldline metric $\gamma_{\tau\tau}$, we could construct an invariant action by covariantizing the action above with respect to the worldline metric as follows:

  $\int \dot{X}^2 + \int m^2 \to \int \sqrt{\gamma}\gamma^{\tau\tau}\dot{X}\dot{X} + \sqrt{\gamma}m^2$

  In this one dimensional setup, $\det(\gamma_{\mu\nu}) = g_{\tau\tau}$ *We can see that when we introduced η,*
  it behaves exactly as the square root of γ, namely $\eta = \sqrt{\gamma_{\tau\tau}}$. Since η is dynamical, this means that we have introduced dynamical gravity on the worldline of the particle.

We can use the fact that η transforms in this way to choose τ so that $\eta = 1$. This simplifies the dynamics.

Then

$$S_f = -\frac{1}{2}\int d\tau \left[\left(\frac{dx^\mu}{d\tau}\right)^2 + m^2\right] \tag{21}$$

From here, the equations of motion are

$$\ddot{x}^\mu = 0 \tag{22}$$

which again implies that particles move in straight lines.

Again, we can define the conjugate momenta to the $x^\mu(\tau)$ and Legendre transform to find that

$$H_\tau = \frac{1}{2}(m^2 + p^\mu p_\mu) = \frac{1}{2}(m^2 + \vec{p}^2 - p_0^2) \tag{23}$$

and it would seem that H_τ has an unbounded spectrum.

However, S_f is not equivalent to S, because in S we also have to impose the e.o.m. of η. The equations of motion of η imply that

$$H_\tau = 0 \tag{24}$$

To quantize we need to impose $H_\tau = 0$ as a constraint. Naively, this implies that there is no dynamics. In practice, we need to write wave functions that satisfy $H_\tau\psi = 0$ This is called the Wheeler-De Witt equation: Happens for any theory that is reparametrization invariant in a covariant approach.

This implies that ψ satisfies the Klein-Gordon equation.

From here one can build a unitary irreducible representation of the Lorentz group:

$$\exp(ik_\mu x^\mu)$$

for $k_0 > 0$, these are particles (moving forward in time). [1]

Since k_0 is a function of the 3-vector $\vec{k}$, the states are characterized by their three momentum,

$$|\vec{k}\rangle$$

Notice that this matches the set of one particle states in a free scalar field theory for a field of mass m. This is with the identification

$$|\vec{k}\rangle \simeq a^\dagger_{\vec{k}}|0\rangle \tag{25}$$

Notice also that the inverse operator of the Hamiltonian constraint is the space-time propagator for quantum field theory of scalar particles. The fact that we get the Klein-Gordon equation here, is what makes the connection to quantum field theory possible.

4 Coupling to an EM field

Again, one can consider a relativistic particle moving in an EM field. The EM field is determined by a one form potential $A = A_\mu dx^\mu$, by $F = dA$. From here we can construct a reparametrization invariant quantity. (In differential form language, it is the integral of the pull-back of A on the worldline of the particle, namely $\int A^*$), where

$$A^*_\alpha d\tau^\alpha = A_\mu \frac{\partial x^\mu}{\partial \tau^\alpha} d\tau^\alpha$$

The action for the relativistic particle needs to be modified as follows

$$S_e = S_R \pm e \int d\tau A_\mu \frac{dx^\mu}{d\tau} \tag{26}$$

Notice that the last term is (obviously) reparametrization invariant, and the e.o.m are invariant under gauge transformations of $A_\mu \to A_\mu + \partial_\mu f$, because the action changes by a total derivative. Above, the sign determines if

[1] This requires introducing a non-obvious inner product from the point of view of the space of L^2 functions on R^d. Notice that in order to solve the constraint, the fact that the spectrum of H is continuous at zero is of the essence.

the particle is positively or negatively charged to the EM field, with absolute value of the charge equal to e.

5 Coupling to gravity

It is straightforward to couple the point particle to gravity. All we need to do is change the proper time to accomodate a different geometry.

If we have a metric $g_{\mu\nu}$ on spacetime, then the action of the point particle will be

$$S = -m \int ds = -m \int \sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\mu}} \tag{27}$$

If we had a lagrangian for a non-relativistic particle in an Euclidean geometry of the form

$$L = \frac{1}{2M} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \tag{28}$$

This action would be called a non-linear sigma model action.

If one follows the covariant prescription for solving the relativistic particle dynamics, then one finds the non-linear sigma model action for the metric $g_{\mu\nu}$ of spacetime.