Week 10

Reading material from the books

- *Polchinski, chapter 10*
- *Becker, Becker, Schwartz, Chapter 4*
- *Green, Schwartz, Witten, chapter 4,5*

1 Worldsheet Supersymmetry

So far we have talked about the theory of the bosonic string, and we have found that such a theory is unphysical because it does not predict spacetime fermions. These are observed in nature.

We have also found that in considering the theory of free fermions on the worldsheet there is a possibility of having spinor representations if we have zero modes on the circle, because the relation of the zero modes

$$\{\psi_0^i, \psi_0^j\} \sim \delta^{ij} \tag{1}$$

is the same algebra of gamma matrices.

However, if we just add fermions on the worldsheet, the directions associated to the indices i, j would not have anything to do with spacetime, but instead they would be associated to some internal symmetry.

What we want is to have the i, j indices of fermions to correspond to $\mu\nu$, the spacetime indices.

For this we need a symmetry that can change the bosonic fields X^μ, into fermions ψ^μ and viceversa. Such a symmetry is called supersymmetry.

This should close on the equations of motion.

Since the fermions are holomorphic, the fermions should transform into a holomorphic object made of X. IN particular, ∂X^μ fits the bill.

Putting it together, we should have

$$\delta X^\mu \sim \psi^\mu \tag{2}$$

$$\delta\psi^\mu \sim \partial X^\mu \tag{3}$$

This means that the associated supersymmetry variation has dimension one half, and the parameter that is associated to it has dimension $(-1/2)$ (it is like a half vector).

Also notice that a double supersymmetry variation results in a translation. Thus, this is a symmetry that enlarges the Lorentz group (in this case in two dimensions, but the generalization to higher dimensions has the same property).

The associated current that generates those transformations is given by

$$G \sim \psi^\mu \partial X^\mu \tag{4}$$

and something similar for the anti-holomorphic fields.

The idea is now to require that instead of just gauging $T = 0$, in supersymetric theories we will also need to set $G = 0$, because the two mix with each other.

To realize this in a BRST context, we will need a pair of ghosts that give us the extra condition $G = 0$. These are called $\beta\gamma$ ghosts. The γ will end up having the same properties of c, and β will be the associated fermionic partner of b.

In the end, we also get a free theory.

The BRST operator should include a term of the form

$$\oint \gamma G_{matter} dz \tag{5}$$

For this to have conformal dimension equal to one, we need γ TO HAVE DIMENSION $(-1/2)$. Similarly, β should have dimension $3/2$, and it should roughly be the case that

$$[Q_{BRST}, \beta] \sim G \tag{6}$$

so that $G = 0$ is imposed via the physical state constraint that $[Q_{BRST}, V] = 0$.

Also, Q_{BRST} is fermionic, but G is already fermionic, so the $\beta\gamma$ ghosts are bosons (they have the opposite statistics of the corresponding symmetry).

These should have the OPE's

$$\beta(z)\gamma(w) \sim \frac{1}{z-w} \tag{7}$$

$$\gamma(z)\beta(w) \sim \frac{-1}{z-w} \tag{8}$$

while the other OPE's are regular (there is no contractions). The extra minus sign in the OPE's relative to a standard fermionic system is what makes these into bosons.

It is easy to show that if we have the standard tachyon vertex operator $V \sim: \exp(ikX)$:, that the BRST part proportional to c gives us that this has to be an operator of dimension 1. We should also have that

$$\oint \frac{dz}{2\pi i}\beta(z)G(z)V(0) = 0 \tag{9}$$

but one can show explicitly that this is not the case. Instead on the right hand side we get $i\beta(0)k \cdot \psi^\mu \exp(ikX)$.

Thus, we find that the standard tachyon is not an allowed vertex operator. This help us towards eliminating the tachyon.

At level one half (we now have fermionic oscillators, so their canonical dimension is one half), we find that we should need a vertex operator of the form

$$\epsilon_\mu \psi^\mu \exp(ikX) \tag{10}$$

The same restrictions as before give us that $\exp(iKX)$ has to have dimension one half, so that

$$\frac{k^2}{2} = 1/2 = -m^2 \tag{11}$$

and we find that these states would also be tachyonic, but instead of having mass 2, they would have mass equal to one (one half of the standard tachyon mass).

The additional constraint imposed by G, would give us that (for the term with two fermions) $k_\mu \epsilon_\nu - k_\nu \epsilon_\mu = 0$. This is, we need to have ϵ to be proportional to k.

Once we do this, we find that for the term with no fermions, we get that it is proportional to

$$\beta(k_\mu \partial X^\mu \exp(ikX)) \tag{12}$$

and the matter part is a total derivative, so for integrated operators this is 'trivial'. This indicates that to fix this and get a total derivative, we need to end up adding some extra ghost pieces of dimension one half.

At level one, we should use operators of the form $\partial X^\mu + \psi^\rho \psi^\sigma$.

The standard OPE with T tells us that this should be a dimension one operator, so that $k^2 = 0$.

Again, the terms with one and three fermions asks that the combination that shows up is roughly

$$(\partial X^\mu + ik_\nu \psi^\nu \psi^\mu)\exp(ikX) \tag{13}$$

We can also check that the OPE of two of these has a zero residue on shell for the standard tachyon. This results from the extra contractions of the ψ, which also contribute a double pole with powers of k. Also, since the "half tachyon" has an odd number of ψ fields, it does not show up in the OPE of the standard massless vertex operators.

There is a hope that all tachyons are eliminated from scattering amplitudes if everything is done carefully. All this argument has been for the NS sector.

We also need to examine the Ramond sector. However, here things get more messy. This is because we now have periodic boundary conditions for the ψ, and therefore also for the $\beta\gamma$. More troublesome is the fact that now β, γ have zero modes, and they are bosons, so it is not something that can be relegated to an extra simple degeneracy of the ground state.

At this stage all of this looks rather complicated, and it behooves us to ask if there is an easy way to understand why the vertex operators take the form that they do. This is what we will do now.

2 Superspace

For states in the NS sector, there is no branch cut associated to the supercurrent OPE with the field, so it corresponds to a sector where supersymmetry is not broken. In this setup,, $G_{matter} \cdot V_{matter}$ will pick the single pole term contribution, so it is associated to

$$\oint G_{matter} V \tag{14}$$

These contour integrals are global supersymmetry transformations.

Usually, supersymmetry transformations commute to translations. this is because the supersymmetry algebra is technically of the form

$$\{Q, Q\} \sim P \tag{15}$$

The momentum operator P is usually represented (up to factors of i, etc) by

$$\frac{\partial}{\partial z} = \partial_z \tag{16}$$

It would be nice if Q could also be described as a vector field in a space, roughly, imagine that we could think of Q as being of the form

$$Q \sim \partial_\theta \tag{17}$$

To get the statistics of Q correctly, θ should be a Grasman variable (a-number).

Also, one finds that the Q defined above does not square to P. Instead, a corrected vector field that gives the right algebra for the supersymmetry is

$$Q \sim \partial_\theta + \theta \partial_z \tag{18}$$

We fix the constant above in the algebra to be equal to two.

The idea is that these vector fields give us a linear realization of supersymmetry (representation of supersymmetry) on a set of functions.

These vector fields should act on a set of functions of z, θ. Physical fields whose realization of supersymmetry is linear, can be written so that they depend on z, θ. These are called superfields, and the set of variables z, θ are called superspace. Fields will naturally be functions of z, θ and should admit a Taylor series expansion.

Since θ is a Grassman variables, expansions in θ terminate, and we can write superfields as follows

$$X^\mu(z, \theta) = X^\mu(z, 0) + \theta \partial_\theta X^\mu(z, \theta)|_{\theta=0} \tag{19}$$

or, equivalently

$$X^\mu(z, \theta) = X^\mu(z) + \theta \psi^\mu(z) \tag{20}$$

If we multiply many superfields, the fact that Q acts on the Hilbert space of fields by graded commutators gives a Leibnitz rule. This is automatically realized in superspace, so the product of superfields is always another superfield.

On ordinary space, we are also allowed to take derivatives and produce new fields. Because derivatives commute with each other, this derivative action gives again a linear realization of translation invariance.

One can generalize this to tensors by including rotations and symbols of the type dx.

In the supersymmetric setup, we ned to find derivative operators that commute with supersymmetry.

It's easy to show that ∂_z does that, and it is just like taking ordinary derivatives.

Now, it is also easy to show that there is a fermionic derivative that commutes with supersymmetry:

$$D_\theta = \partial_\theta - \theta \partial_z \tag{21}$$

The proof is simple, just expand the expression

$$\{Q, D_\theta\} \tag{22}$$

This is a covariant derivative with respect to supersymmetry. We are going to abuse notation and we will call $D_\theta = D$.

In the end we are also going to have supersymmetry for right movers. Our superfield will be of the form

$$X^\mu(z, \bar{z}, \theta, \bar{\theta}) = X^\mu(z, \bar{z}) + \theta\psi^\mu + \bar{\theta}\bar{\psi}^\mu + \theta\bar{\theta}F \tag{23}$$

The field F is introduced by realizing supersymmetry linearly. It is called an auxiliary field. This is common in superspace.

For the right movers, we will introduce $D_{\bar{\theta}}$ in a similar fashion, which we will call $\bar{D}$.

We are interested now in producing supersymmetric actions.

These should be of the form

$$\int d^2z d^2\theta G \sim \int d^2z F_G \tag{24}$$

where G is a superfield lagrangian density, built out of superfields and their covariant derivatives. remember that integration over Grassman variable is of the form

$$\int d\theta 1 = 0 \quad \int d\theta\theta = 1 \tag{25}$$

so that for practical purposes

$$\int d\theta \sim \partial_\theta \tag{26}$$

The integral over z guarantees translation invariance, while the integral $\int d\theta$ guarantees "super-translation invariance".

The idea is that for any superfield G, it will have an expansion of the form given above.

The action of supersymmetry on the field guarantees that

$$Q \cdot F_G = \partial_z \bar{\psi}_G \tag{27}$$

while

$$\bar{Q} \cdot F_G = \bar{\partial}_z \psi_G \tag{28}$$

so under supersymmetries the Lagrangian density transforms by a total derivative.

To get a standard action for the field $X^\mu(z,\bar{z})$, we need the following supersymmetric lagrangian density

$$\frac{1}{2}\int d^2\theta D X^\mu \bar{D} X^\mu \tag{29}$$

If we integrate over the θ variables, we find that the lagrangian density takes the usual form for $X^\mu, \psi^\mu, \bar{\psi}^\mu$.

However, we also get a term of the form $\int \frac{1}{2}F^2$.

Notice that this has no kinetic term for F, so F can be systematically integrated out and removed from the lagrangian. Indeed, one can show that F will only appear linearly in the action if we also include an interaction potential.

Putting supersymmetry on a general target space, we have to use the following action

$$\int d^2\theta g_{\mu\nu}(X) D X^\mu \bar{D} X^\mu \tag{30}$$

Notice that expanding the action above, we find terms that involve four fermion interactions

$$\partial_\rho \partial_\sigma g_{\mu\nu} \psi^\rho \bar{\psi}^\sigma \bar{\psi}^\mu \psi^\nu \tag{31}$$

This actually gets covariantized to the target space curvature tensor $R_{\rho\sigma\mu\nu}$.

Also, just like vertex operators are integrated over the worldsheet coordinates in order to have manifest invariance under diffeomorphisms, in superspace one should have to do an integral over the super-coordinates as well.

If we write everything in terms of superfields, this makes sense, and suggests that vertex operators are of the form

$$\int d^2\theta \int d^2 z V(z,\theta,\bar{\theta}) \tag{32}$$

The natural exponential is the exponential of the superfield, so we would have

$$V \sim \exp(ikX + ik\theta\psi + ik\bar{\theta}\bar{\psi} + \theta\bar{\theta}F) \tag{33}$$

Also the constraints would require that V is a superconformal primary (this is a conformal primary which is also annihilated by the negative energy modes of G).

If we ignore the auxiliary fields, we get that we need insertions of (let us consider only left movers)

$$\int d\theta \exp(ikX)(1 + i\theta k \cdot \psi) \sim \exp(ikX) k \cdot \psi \tag{34}$$

just as we found before.

Similarly, we would consider operators of the form

$$\begin{aligned} \int d\theta D X^\mu \exp(ikX)(1 + i\theta k \cdot \psi) &\sim \int d\theta(\psi^\mu + \theta \partial X^\mu)(1 + i\theta k \cdot \psi) \quad (35) \\ &\sim (\partial X^\mu + \psi^\mu i k \cdot \psi) \exp(ikX) \quad (36) \end{aligned}$$

and we would also get that $\epsilon \cdot k = 0$ and $\epsilon \to \epsilon + k$ is the BRST cohomology (then the vertex operator is a total superderivative). So that we still have only transverse directions to the lightcone.

This explains why the vertex operators we were getting have the form they have. We see that this has ended up simplifying the system considerably (at least as far as the NS sector is conerned).